

Math 115 Survey of Calculus I Notes

Fall 2013

Intended for use with Calculus with Applications; 10th Edition
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Section R.6: Exponents

Exponent: $a^n = a \cdot a \cdot a \cdot \dots \cdot a$, n times, where a is the base.

$$a^0 = 1, \text{ where } a \neq 0.$$

$$a^{-n} = \frac{1}{a^n}$$

Example 1: Simplify:

A) 6^0

B) 3^{-2}

Properties of Exponents:

1) $a^m a^n = a^{m+n}$

2) $\frac{a^m}{a^n} = a^{m-n}$

3) $(a^m)^n = a^{mn}$

4) $(ab)^n = a^n b^n$

5) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example 1: Simplify:

A) $2^4 \cdot 2^6$

B) $\frac{3^{14}}{3^6}$

C) $(2x^3)^4$

D) $\left(\frac{x^2}{y^3}\right)^6$

Rational Exponents: $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$

Example 2: Rewrite and simplify:

A) $256^{1/4}$

B) $27^{2/3}$

Example 3: Simplify:

A) $\frac{y^{1/3} y^{5/3}}{y^3}$

B) $m^{2/3}(m^{7/3} + 2m^{1/3})$

Section R.7: Radicals

Example 1: Simplify:

A) $\sqrt[4]{16}$

B) $\sqrt[5]{-32}$

Properties of Radicals:

1. $(\sqrt[n]{a})^n = a$

2. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$

3. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Example 2: Simplify:

A) $\sqrt{48}$

B) $\sqrt{1,000}$

C) $\sqrt{128}$

Example 3: Simplify: $2\sqrt{18} - 5\sqrt{32}$

You can only add or subtract radicals if the number under the radical is the same.

Example 4: Simplify: $\sqrt{x^5}\sqrt[3]{x^5}$

Example 5: Rationalize the denominator:

A) $\frac{4}{\sqrt{3}}$

The conjugate of $a + b$ is $a - b$.

B) $\frac{1}{1-\sqrt{2}}$

Example 6: Rationalize the numerator: $\frac{\sqrt{x}-3}{x-9}$

Section 1.1: Slopes and Equations of Lines

Slope: the slope of the line containing the points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{\text{change in } y}{\text{change in } x} \text{ or } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1: Find the slope of the line between the two points:

A) $(7, 6)$ and $(-4, 5)$

B) $(5, -3)$ and $(-2, -3)$

C) $(2, -4)$ and $(2, 3)$

Slope-Intercept Form of a Line: $y = mx + b$, where m is the slope and b is the y-intercept.

Point-Slope Form of a Line: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line.

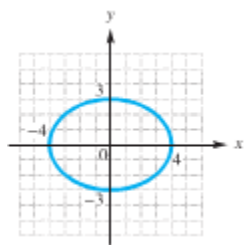
Example 2: Find the equation of the line through $(5, 4)$ and $(-10, -2)$.

Section 2.1: Properties of Functions

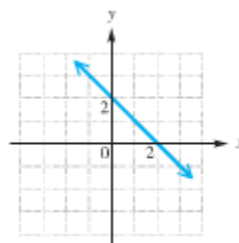
Function: A rule that assigns to each element from one set exactly one element from another set.

Vertical Line Test: If a vertical line intersects a graph in more than one place, the graph is not a function.

Example 2: Is it a function?



A)



B)

Domain: the x-values – can't divide by zero, can't take square root of a negative

Range: the y-values – use the graph

Example 3: Find domain and range:

A) $\{(1, 3), (4, 7), (12, 18)\}$

B) $y = x^2$

Example 4: Find the domain:

A) $f(x) = \frac{x+7}{x-3}$

B) $f(x) = \sqrt{x-2}$

Example 5: Use the function $f(x) = -x^2 + 4x + 6$ to evaluate:

A) $f(a-1)$

B) $\frac{f(x+h) - f(x)}{h}$

Section 2.2: Quadratic Functions

Quadratic Function: of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Example 1: Graph: $f(x) = x^2 - 4$. Find the intercepts and the vertex.

Example 2: Graph $f(x) = -3x^2 - 6x + 4$. Find the intercepts and the vertex.

Section 2.3: Polynomial and Rational Functions

Polynomial Function: of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Example 1: Graph: $f(x) = 8x^3 - 12x^2 + 2x + 1$

Rational Function: of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and $q(x) \neq 0$.

Vertical Asymptote: where the denominator equals zero.

Example 2: Graph and find vertical asymptotes: $f(x) = \frac{x+2}{x-1}$

Example 3: Graph and find vertical asymptotes: $f(x) = \frac{x^2+3x+2}{x+1}$

Section 2.4: Exponential Functions

Exponential Function: of the form $f(x) = a^x$, where a is the base, and $a > 0$, $a \neq 0$.

Example 1: Graph: $f(x) = 2^x$

Exponential Property: if $a^x = a^y$, then $x = y$.

Example 2: Solve: $9^x = 27$

Example 3: Solve: $32^{2x-1} = 128^{x+3}$

The Number e: the irrational number $e \approx 2.718$

Example 4: Graph: $f(x) = e^x$

Section 2.5: Logarithm Functions

Logarithm: $y = \log_a x$ "log base a of x" means $a^y = x$, where $a > 0$, $a \neq 1$, and $x > 0$.

Example 1: Rewrite:

A) $2^3 = 8$

B) $\log_5 25 = 2$

Example 2: Solve:

$$\log_x 16 = 4$$

Rewrite as an exponential then solve.

Example 3: Solve:

$$\log_4 x = \frac{3}{2}$$

Example 4: Solve:

$$\log_8 4 = x$$

Common Log: $\log x = \log_{10} x$

Natural Log: $\ln x = \log_e x$

Example 5: Evaluate: $\log 12$

Example 6: Graph: $f(x) = \ln x$

Log Properties:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a xy = \log_a x + \log_a y$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$
5. $\log_a x^r = r \log_a x$
6. $\log_a a^r = r$

Example 7: Use Log Properties to simplify:

A) $\log_{17} 1$

B) $\log_{30} 30$

C) $\log_2 3x$

D) $\log_3 4 + \log_3 y$

E) $\log_4 \frac{3}{x}$

F) $-\log_5 y + \log_5 10$

G) $\log_6 x^2$

H) $4 \log_7 y$

I) $\log 10^x$

Example 8: Expand: $\log_b \frac{a^7 c^8}{b^9 d^{10}}$

Example 9: Write as a single log: $2 \log_3 x - 3 \log_3 y + 4 \log_3 w - 5 \log_3 z$

Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$, where b is a number we choose, usually 10 or e.

Example 10: Evaluate: $\log_5 17$

Example 11: Graph: $f(x) = \log_3(x + 1)$

Example 12: Solve: $5^x = 7$

Example 13: Solve: $2^{x+4} = 3^{x-5}$

Example 14: Solve: $\log(x + 5) + \log(x + 2) = 1$

Extra Algebra: Factoring

Example 1: Factor: $3x^2(2x - 7)^4(5x + 8)^9 + 6x(2x - 7)^3(5x + 8)^{10}$

Example 2: Factor: $5x(3x + 10)^7(x + 8)^{-3} + 10x(3x + 10)^6(x + 8)^{-2}$

Extra Algebra: Simplifying

Example 1: Simplify: $\frac{\frac{2}{x} + \frac{3}{x+1}}{2x-7}$

Example 2: Simplify: $\frac{1 + \frac{\sqrt{y}}{2\sqrt{x}}}{2y - \frac{\sqrt{x}}{2\sqrt{y}}}$

Example 3: Simplify:
$$\frac{(\ln(x^2 + 3) + 1)\left[\frac{2x}{x^2 + 3} + 1\right] - (\ln(x^2 + 3) + x)\left[\frac{2x}{x^2 + 3}\right]}{(\ln(x^2 + 3) + 1)^2}$$

Extra Algebra: Solving:

Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 1: Solve: $x^2 + 4x + 6 = 0$

Example 2: Solve and find domain: $\frac{x+7}{2x-5} = 0$

Example 3: Solve and find domain: $\frac{x3^x(\ln 3) - 3^x}{x^2} = 0$

Section 3.1: Limits

Example 1: For the function $f(x) = \frac{x^2-4}{x-2}$, what is the value for $f(x)$ when x gets very close to 1?

x	.9	.99	1	1.01	1.1
f(x)					

Limit Notation: $\lim_{x \rightarrow a} f(x)$ “the limit as x approaches a of $f(x)$ ”

$\lim_{x \rightarrow a^-} f(x)$ “the limit as x approaches a from the left”

$\lim_{x \rightarrow a^+} f(x)$ “the limit as x approaches a from the right”

Limit of a Function: f is a function, a and L are real numbers. If:

1. as x takes values closer and closer (but not equal) to a on both sides of a , the corresponding values of $f(x)$ get closer and closer (and maybe equal) to L ;

2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a ;

Then L is the limit of $f(x)$ as x approaches a . $\lim_{x \rightarrow a} f(x) = L$

Example 2: Find the limit: $\lim_{x \rightarrow 1} \frac{x^2+4}{x-2}$

Example 3: Find the limit: $\lim_{x \rightarrow -2} \frac{3x+2}{2x+4}$

Existence of Limits: The limit of f as x approaches a may not exist.

1. If $f(x)$ becomes infinitely large (positive or negative) as x approaches a from either side, $\lim_{x \rightarrow a} f(x) = \pm\infty$, depending. (Graph to see which.)

2. If $f(x)$ becomes infinitely large (positive) as x approaches a on one side and infinitely large (negative) on the other side, then $\lim_{x \rightarrow a} f(x) = DNE$ (does not exist)

3. If $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = M$ and $L \neq M$, then $\lim_{x \rightarrow a} f(x) = DNE$

Example 4: $\lim_{x \rightarrow 0} \frac{1}{x^2}$

Example 5: $\lim_{x \rightarrow 0} \frac{1}{x}$

Example 6: $\lim_{x \rightarrow 0} \frac{|x|}{x}$

Rules for Limits: Given that a, A, and B are real numbers, $\lim_{x \rightarrow a} f(x) = A$ and $\lim_{x \rightarrow a} g(x) = B$.

1. if k is a constant, then $\lim_{x \rightarrow a} k = k$ and $\lim_{x \rightarrow a} k \cdot f(x) = k \cdot [\lim_{x \rightarrow a} f(x)] = kA$
2. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = A \pm B$
3. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = A \cdot B$
4. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{A}{B}$, provided $B \neq 0$.
5. If p(x) is a polynomial, then $\lim_{x \rightarrow a} p(x) = p(a)$
6. for any real number k, $\lim_{x \rightarrow a} [f(x)]^k = [\lim_{x \rightarrow a} f(x)]^k = A^k$
7. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ if $f(x) = g(x)$ for all x
8. for any real number b > 0, $\lim_{x \rightarrow a} b^{f(x)} = b^{\lim_{x \rightarrow a} f(x)} = b^A$
9. for any real number b such that $0 < b < 1$ or $1 < b$, then $\lim_{x \rightarrow a} [\log_b f(x)] = \log_b [\lim_{x \rightarrow a} f(x)] = \log_b A$, if $A > 0$

Example 7: $\lim_{x \rightarrow 3} (3x - 5)$

Example 8: $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

Example 9: $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

Limits of Infinity: for any positive real number n, $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0$

Example 10: $\lim_{x \rightarrow \infty} \frac{8x + 6}{3x - 1}$

Example 11: $\lim_{x \rightarrow \infty} \frac{3x+2}{4x^3-1}$

Example 12: $\lim_{x \rightarrow \infty} \frac{3x^2+2}{4x-3}$

Section 3.2: Continuity

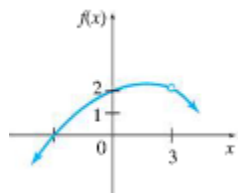
Continuity (informal definition): if you can draw the graph without picking up your pencil from the paper.

Continuity at $x = c$: the function f is continuous at c if:

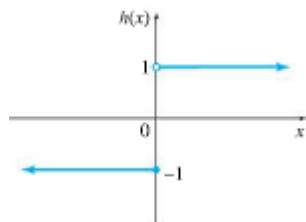
1. $f(c)$ is defined.
2. $\lim_{x \rightarrow c} f(x)$ exists.
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Example 1: Say why the function is discontinuous at the point:

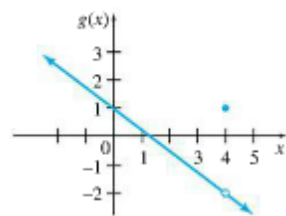
A) at $x = 3$:



B) at $x = 0$:



C) at $x = 4$:



Continuity on a Closed Interval: the function f is continuous on the closed interval $[a, b]$ if:

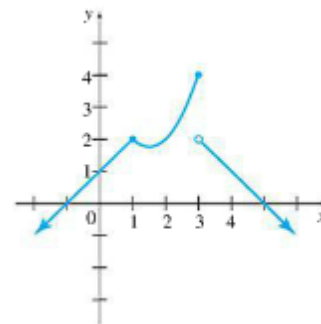
1. f is continuous on the open interval (a, b) .
2. f is continuous from the right at $x = a$.
3. f is continuous from the left at $x = b$.

Example 2: $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$

Type of function	Polynomial	Rational	Root	Exponential	Logarithmic
Where it's continuous	All x	Denominator $\neq 0$	No square root of a negative	All x	Only take a log of a positive

Example 3: Find all values where the function is discontinuous:

$$f(x) = \begin{cases} x + 1, & \text{if } x < 1 \\ x^2 - 3x + 4, & \text{if } 1 \leq x \leq 3 \\ 5 - x, & \text{if } x > 3 \end{cases}$$



Section 3.3: Rates of Change

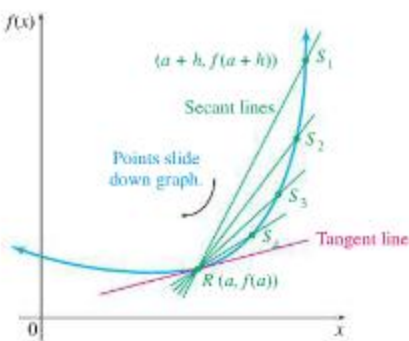
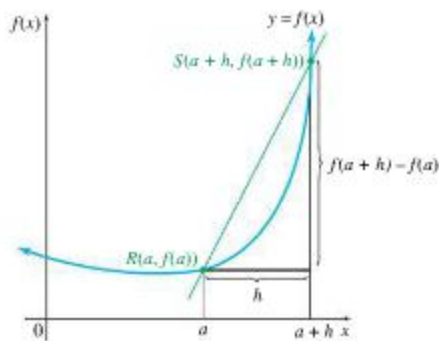
Instantaneous Rate of Change: of the function f when $x = a$ is $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

Example 2: Find the instantaneous rate of change for $f(x) = 2x^2 - 5x + 40$ at $x = 2$.

Example 3: Find the instantaneous rate of change of $f(x) = 3x^2 + 4x - 7$ at $x = 5$.

Section 3.4: Definition of the Derivative

Slope of the Tangent Line: The tangent of $y = f(x)$ at the point $(a, f(a))$ is $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ which is the instantaneous rate of change and the derivative at that point.



Derivative: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ read "f prime of x"

Example 1: Find f' for the function $f(x) = x^2 - 5$

Example 2: Find f' for the function $f(x) = 2x^3 + 4x$

Example 3: Find f' for the function $f(x) = \frac{4}{x}$

Example 4: Find f' for the function $f(x) = \sqrt{x} + 40$

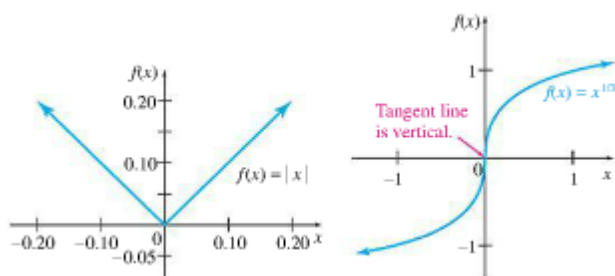
Example 5: Find $f'(2)$ for the function $f(x) = 4x^2 - 7x + 2$

Equation of the Tangent Line: The line tangent to $y = f(x)$ at the point $(x_1, f(x_1))$ is $y - f(x_1) = f'(x_1)(x - x_1)$

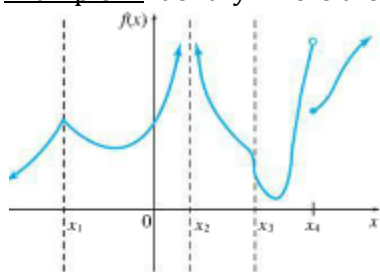
Example 6: Find the equation of the tangent line to $f(x) = \frac{-6}{x}$ at $x = 4$.

Existence of the Derivative: The derivative of f exists if all of the following are true:

1. f is continuous
2. f is smooth
3. f does not have a vertical tangent line



Example 7: Identify where the function does not have a derivative:



Section 4.1: Techniques for Finding the Derivative

Notation: $f'(x)$, $\frac{dy}{dx}$ (derivative of y with respect to x), $\frac{d}{dx}[f(x)]$, D_x

Constant Rule: if k is a constant, $\frac{d}{dx}[k] = 0$

Example 1: $\frac{d}{dx}[9]$

Example 2: $\frac{d}{dx}[\sqrt{3}\pi]$

Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$

Example 3: $\frac{d}{dx}[x^4]$

Example 4: $\frac{d}{dx}\left[\frac{1}{x^6}\right]$

Example 5: $\frac{d}{dx}[\sqrt{x}]$

Example 6: $\frac{d}{dx}[x]$

Constant Times a Function: if k is a constant, $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}[f(x)] = k \cdot f'(x)$

Example 7: $\frac{d}{dx}[2x^3]$

Example 8: $\frac{d}{dx}\left[\frac{5}{x}\right]$

Example 9: $\frac{d}{dx} \left[\frac{3}{4} \sqrt[3]{x} \right]$

Sum or Difference Rule: $\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)] = f'(x) \pm g'(x)$

Example 10: $\frac{d}{dx} [5x^3 + 2x - 12]$

Example 11: $\frac{d}{dx} \left[12x^4 - 6\sqrt{x} + \frac{5}{x} \right]$

Example 12: $\frac{d}{dx} \left[\frac{x^3 + 3\sqrt{x}}{x} \right]$

Example 13: For the function $f(x) = 4x^3 - 17x^2 + 18x - 21$, find $f'(2)$.

Section 4.2: Derivatives of Products and Quotients

Product Rule: $\frac{d}{dx} [f(x)g(x)] = f(x) \cdot \frac{d}{dx} [g(x)] + g(x) \cdot \frac{d}{dx} [f(x)]$

Example 1: $\frac{d}{dx} [(2x + 3)(3x^2)]$

Example 2: $\frac{d}{dx} [(\sqrt{x} + 3)(x^2 - 5x)]$

Quotient Rule: $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{(g(x))^2}$ or $\frac{d}{dx} \left[\frac{hi}{lo} \right] = \frac{lo \frac{dhi}{dx} - hi \frac{dlo}{dx}}{lo^2}$

Example 3: $\frac{d}{dx} \left[\frac{2x-1}{4x+3} \right]$

Example 4: $\frac{d}{dx} \left[\frac{3x^2+12}{2x+1} \right]$

Section 4.3: Chain Rule

Chain Rule: $\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$ or $\frac{d}{dx} [f(x)^n] = n f(x)^{n-1} f'(x)$

Example 1: $\frac{d}{dx} [(x^2 + 5x)^8]$

Example 2: $\frac{d}{dx} [(3x^3 + 7)^4]$

Example 3: $\frac{d}{dx} [4x(3x + 5)^5]$

Example 4: $\frac{d}{dx} [(2x + 3)^7 (5x - 6)^8]$

Example 5: $\frac{d}{dx} \left[\frac{(3x+2)^7}{x-1} \right]$

Example 6: $\frac{d}{dx} \left[\frac{(4x-3)^5}{6x+5} \right]$

Section 4.4: Derivatives of Exponential FunctionsExponential Rules:

$$1. \frac{d}{dx} [e^x] = e^x$$

$$2. \frac{d}{dx} [e^{f(x)}] = e^{f(x)} \cdot \frac{d}{dx} [f(x)]$$

$$3. \frac{d}{dx} [a^x] = (\ln a) \cdot a^x$$

$$4. \frac{d}{dx} [a^{f(x)}] = (\ln a) \cdot a^{f(x)} \cdot \frac{d}{dx} [f(x)]$$

Example 1: $\frac{d}{dx} [3e^x]$

Example 2: $\frac{d}{dx} [e^{5x}]$

Example 3: $\frac{d}{dx} [2^x]$

Example 4: $\frac{d}{dx} [10^{2x+3}]$

Example 5: $\frac{d}{dx} \left[\frac{10}{1+e^{-x}} \right]$

Example 6: $\frac{d}{dx} \left[e^{x^2+1} \sqrt{5x+2} \right]$

Example 7: $\frac{d}{dx} \left[\frac{x^2 e^{2x}}{x + e^{3x}} \right]$

Section 4.5: Derivatives of Logarithmic FunctionsLog Rules:

$$13. \frac{d}{dx} [\ln|x|] = \frac{1}{x}$$

$$14. \frac{d}{dx} [\ln|f(x)|] = \frac{\frac{d}{dx}[f(x)]}{f(x)}$$

$$15. \frac{d}{dx} [\log_a|x|] = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$16. \frac{d}{dx} [\log_a|f(x)|] = \frac{1}{\ln a} \cdot \frac{\frac{d}{dx}[f(x)]}{f(x)}$$

Example 1: $\frac{d}{dx} [4 \ln x]$

Example 2: $\frac{d}{dx} [\ln 6x]$

Example 3: $\frac{d}{dx} [\ln(x^2 + 1)]$

Example 4: $\frac{d}{dx} [\log x]$

Example 5: $\frac{d}{dx} [\log_2(3x^2 - 4x)]$

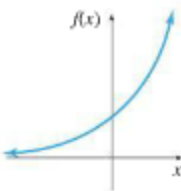
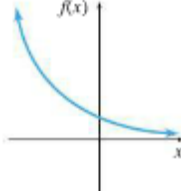
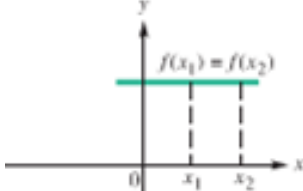
Example 6: $\frac{d}{dx} [3x \ln x^2]$

Example 7: $\frac{d}{dx} [e^x \ln x]$

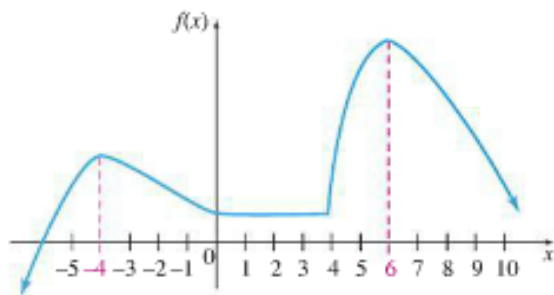
Example 8: $\frac{d}{dx} \left[\frac{\log_8(x^{3/2}+1)}{x} \right]$

Example 9: $\frac{d}{dx} \left[\frac{\ln(x^2+1) + x}{\ln(x^2+1) + 1} \right]$

Section 5.1: Increasing and Decreasing Functions

Increasing: if $f(x_1) < f(x_2)$ where $x_1 < x_2$	Decreasing: if $f(x_1) > f(x_2)$ where $x_1 < x_2$	Constant: if $f(x_1) = f(x_2)$ for all x
		

Example 1: Find the intervals over which the graph is increasing, decreasing, and constant:



Test for Intervals where $f(x)$ is Increasing or Decreasing: f has a derivative at each point on an open interval.

If $f'(x) > 0$ for all x , f is increasing.

If $f'(x) < 0$ for all x , f is decreasing.

If $f'(x) = 0$ for all x , f is constant.

Critical Number: f has a critical number at c if $f'(c) = 0$ or $f'(c) = \text{DNE}$.

Example 2: Find the intervals over which $f(x) = x^3 + 3x^2 - 9x + 4$ is increasing and decreasing.

Step 1: Find f'

Step 2: Find critical numbers.

Step 3: Make a number line and test intervals.

Example 3: Find the intervals over which $f(x) = \frac{x-1}{x+1}$ is increasing and decreasing.

Section 5.2: Relative Extrema

Relative Extrema: $f(c)$ is a critical number of f .

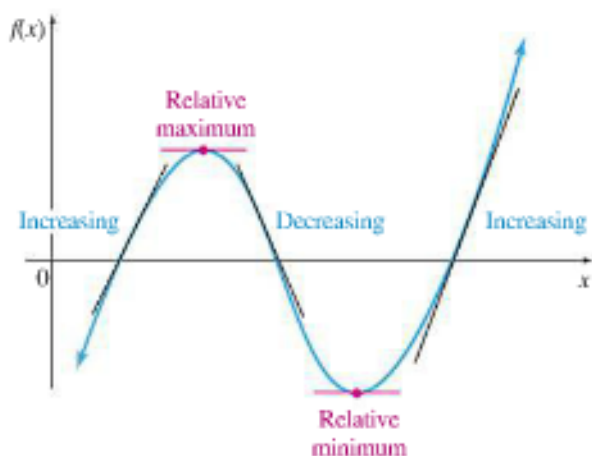
$f(c)$ is a relative maximum if $f(x) \leq f(c)$ for all x on an interval.

$f(c)$ is a relative minimum if $f(x) \geq f(c)$ for all x on an interval.

First Derivative Test: Let c be a critical number for the function f , where f is continuous on the open interval (a, b) and differentiable on the interval (a, b) except maybe at c .

$f(c)$ is a relative maximum if $f'(x) > 0$ when $x < c$ and $f'(x) < 0$ when $x > c$

$f(c)$ is a relative minimum if $f'(x) < 0$ when $x < c$ and $f'(x) > 0$ when $x > c$



Example 1: Find relative extrema for $f(x) = 2x^3 - 3x^2 - 72x + 15$

Example 2: Find relative extrema for $f(x) = 6x^{2/3} - 4x$

Section 5.3: Higher Order Derivatives, Concavity, and the Second Derivative Test

Higher Order Derivatives:

$f''(x)$ – “f double prime” is the second derivative. It is the derivative of f'

Other notation: $\frac{d^2y}{dx^2}, D_{x^2}[f(x)]$

$f'''(x)$ – the third derivative

$f^{(4)}(x)$ – the fourth derivative

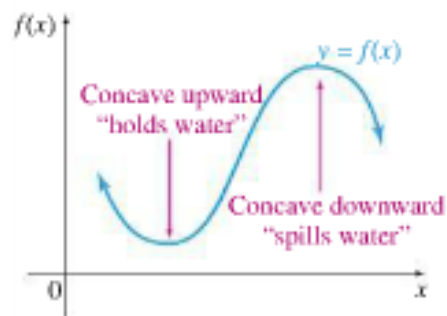
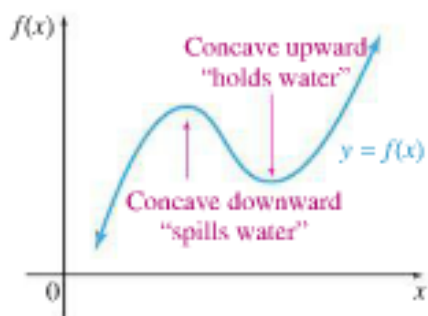
Example 1: Find higher order derivatives for $f(x) = x^4 - 2x^3 + 6x^2 - 9x + 8$

Example 2: Find f'' for $f(x) = (x^2 - 1)^2$

Example 3: Find f'' for $f(x) = 4x(\ln x)$

Example 4: Find f''' for $f(x) = \frac{x}{e^x}$

Concavity:



Test for Concavity: f is a function with f' and f'' existing for all x on the open interval (a, b) .

f is concave up if $f''(x) > 0$ for all x .

f is concave down if $f''(x) < 0$ for all x .

Point of Inflection (POI): $f(c)$ is a point of inflection if $f''(c) = 0$ or $f''(c) = \text{DNE}$. Where concavity changes.

Example 5: Find concavity for $f(x) = x^4 - 8x^3 + 18x^2$

Step 1: Find f'' .

Step 2: Find POI.

Step 3: Perform test for concavity.

Step 4: Check by graphing on calculator.

Example 6: Find concavity for $f(x) = \frac{-2}{x+1}$

Second Derivative Test: f'' exists on an open interval containing c and $f'(c) = 0$.

If $f''(c) > 0$, $f(c)$ is a relative minimum.

If $f''(c) < 0$, $f(c)$ is a relative maximum.

If $f''(c) = 0$ or DNE, the test fails. Use the First Derivative Test and graph.

Example 7: Find relative extrema for $f(x) = 4x^3 + 7x^2 - 10x + 8$

Section 5.4: Curve Sketching

Curve Sketching:

Step 1: Find the domain (identify vertical asymptotes).

Step 2: Find the first and second derivatives.

Step 3: Find the critical numbers ($f'(x) = 0$ or $f'(x) = \text{DNE}$) (also find y-coordinates).

Step 4: Find where the graph is increasing / decreasing. Find relative extrema. (Use either First or Second Derivative Test.)

Step 5: Find the points of inflection ($f''(x) = 0$ or $f''(x) = \text{DNE}$) (also find y-coordinates).

Step 6: Find concavity.

Step 7: Check with calculator. If needed, find other points.

Example 1: Sketch $f(x) = 2x^3 - 3x^2 - 12x + 1$

Example 2: Sketch $f(x) = x + \frac{1}{x}$

Section 6.1: Absolute Extrema

Absolute Minimum or Maximum: f is defined on some interval containing c .

$f(c)$ is the absolute maximum on the interval if $f(x) \leq f(c)$ for all x .

$f(c)$ is the absolute minimum on the interval if $f(x) \geq f(c)$ for all x .

Extreme Value Theorem: A function that is continuous on the closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum.

Find Absolute Extrema:

Step 1: Find all critical numbers (where $f'(x) = 0$ or $f'(x) = \text{DNE}$).

Step 2: Find the y -coordinates for the critical numbers and the endpoints using the TABLE on your calculator.

Step 3: The largest y -value is the absolute maximum. The smallest y -value is the absolute minimum.

Example 1: Find the absolute extrema for $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ over $[-2, 3]$.

Example 2: Find the absolute extrema for $f(x) = x^{8/3} - 16x^{2/3}$ over $[-1, 8]$.

Section 6.2: Applications of Extrema

Example 1: Find two nonnegative numbers x and y for which $2x + y = 30$ and xy^2 is maximized.

Example 2: Find the dimensions of the rectangular field of maximum area that can be made from 200 ft of fencing material.

Section 6.4: Implicit Differentiation

Explicit Differentiation: When $y = 3x + 2$ is used to find $\frac{dy}{dx} = 3$. When y or $f(x)$ is by itself. If you can't solve for y or when y isn't by itself, we must use implicit differentiation.

Example 1: Find $\frac{dy}{dx}$ for $x + y = 7$

Example 2: Find $\frac{dy}{dx}$ for $3xy + 4y^2 = 10$

Example 3: Find $\frac{dy}{dx}$ for $x^3 + y^3 = 9xy$

Example 4: Find $\frac{dy}{dx}$ for $x + \sqrt{x}\sqrt{y} = y^2$

Section 6.5: Related Rates

Example 1: x and y are both functions of t , where $xy^2 + y = x^2 + 17$. When $x = 2$, $y = 3$, and $dx/dt = 13$, find dy/dt .

Example 2: A rock is dropped into a lake. Circular ripples spread over the surface of the water, with the radius of each circle increasing at a rate of $3/2$ feet per second. Find the rate of change of the area inside the circle formed by a ripple at the instant the radius is 4 feet.

Example 3: A 50-foot ladder is placed against a building. The base of the ladder is resting on an oil spill, and it slips at a rate of 3 feet per minute. Find the rate of change of the height of the ladder above the ground at the instant when the base of the ladder is 30 feet from the building.

Example 4: A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, what is the rate of change of the volume of the icicle?

Section 6.6: Differentials

Differential: written dx or dy . $dy = f'(x) \cdot dx$ or $dy = f'(x)\Delta x$

Example 1: Find dy for $y = 6x^2$

Example 2: Find dy for $y = 8x^3 + 2x$ when $x = 5$ and $\Delta x = 0.01$

Section 7.1: Antiderivative

Antiderivative: If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$

Example 1: If $F(x) = 10x$, then $F'(x) = 10$, so $F(x) = 10x$ is an antiderivative of $f(x) = 10$.

Example 2: Find an antiderivative of $f(x) = 5x^4$

Integration Constant: If $F(x)$ and $G(x)$ are both antiderivatives of f , then there is a constant C such that $F(x) - G(x) = C$. C is the integration constant and is added to every indefinite integral.

Indefinite Integral: If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, read as “the integral of $f(x) dx$ ”

Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C = \frac{1}{n+1} \cdot x^{n+1}$, when $n \neq -1$

Example 3: $\int x^3 dx$

Example 4: $\int \frac{1}{x^2} dx$

Example 5: $\int \sqrt{y} dy$

Example 6: $\int z dz$

Constant Rule: $\int k dx = kx + C$

Example 7: $\int 17 dx$

Example 8: $\int 8 dz$

Constant Multiple Rule: $\int k \cdot f(x) dx = k \cdot \int f(x) dx$

Example 9: $\int 2x^5 dx$

Example 10: $\int \frac{12}{x^3} dx$

Sum or Difference Rule: $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

Example 11: $\int (3x^2 - 4x + 5) dx$

Example 12: $\int \frac{x^2+1}{\sqrt{x}} dx$

Example 13: $\int (x^2 - 1)^2 dx$

Exponential Integrals:

$$\int e^x dx = e^x + C$$

$$\int e^{kx} dx = \frac{1}{k} \cdot e^{kx} + C$$

$$\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$$

$$\int a^{kx} dx = \frac{1}{k(\ln a)} \cdot a^{kx} + C$$

Example 14: $\int 9e^x dx$

Example 15: $\int e^{9x} dx$

Example 16: $\int 3e^{\frac{5}{4}x} dx$

Example 17: $\int 2^x dx$

Example 18: $\int 3^{-5x} dx$

Integral of x^{-1} : $\int x^{-1} dx = \int \frac{dx}{x} = \ln |x| + C$

Example 19: $\int \frac{4}{x} dx$

Example 20: $\int \left(-\frac{5}{x} + e^{-2x}\right) dx$

Section 7.2: Substitution

Example 1: $\int 6x(3x^2 + 4)^7 dx$

Rule of thumb 1: If you have a quantity raised to a power, let u = that quantity

Let $u = 3x^2 + 4$, then $du = 6x dx$

Example 2: $\int 15x^2(5x^3 + 2)^5 dx$

Example 3: $\int x^2 \sqrt{x^3 + 1} dx$

Example 4: $\int \frac{x+3}{(x^2+6x)^2} dx$

Example 5: $\int \frac{(2x-3)dx}{x^2-3x}$

Rule of thumb 2: If you have division, let u = the denominator.

Example 6: $\int \frac{x^2+2x}{x^3+3x^2} dx$

Example 7: $\int x^2 e^{x^3} dx$

Rule of thumb 3: If you have e raised to an exponent, let u = the exponent.

Example 8: $\int e^{2x+1} dx$

Example 9: $\int x\sqrt{1-x} dx$

Example 10: $\int \frac{e^{2x}}{e^{2x} + 5} dx$

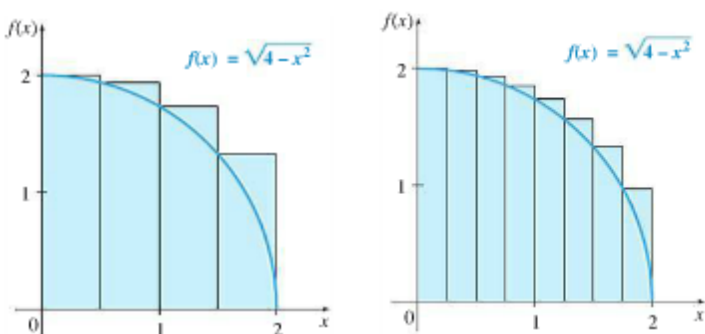
Example 11: $\int \frac{1}{x \ln x} dx$

Section 7.3: Area and the Definite Integral

Definite Integral: If f is defined on the closed interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

“the limit as n approaches infinity of the sum from $i = 1$ to n of $f(x_i)$ times Δx (dx)”



Example 1: Approximate $\int_0^4 2x dx$ by using 4 rectangles of equal width whose heights are the midpoints of each subinterval *and* using geometry.

$$\Delta x \text{ (the width of the rectangles)} = \frac{\text{top endpoint} - \text{bottom endpoints}}{\text{number of rectangles}} = \frac{4-0}{4} = 1$$

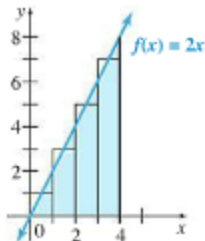
$$\begin{aligned} \int_0^4 2x dx &= \sum_{i=1}^4 f(x_i) \Delta x \\ &= \sum_{i=1}^4 f(x_i)(1) \\ &= f(x_1)(1) + f(x_2)(1) + f(x_3)(1) + f(x_4)(1) \end{aligned}$$

x_1 is midpoint between 0 and 1, so $x_1 = 0.5$

Similarly, $x_2 = 1.5$, $x_3 = 2.5$, and $x_4 = 3.5$.

$$\begin{aligned} &= f(0.5) + f(1.5) + f(2.5) + f(3.5) \\ &= 2(0.5) + 2(1.5) + 2(2.5) + 2(3.5) \\ &= 1 + 3 + 5 + 7 \\ &= 16 \end{aligned}$$

Using geometry, first draw the graph, then use the area of a triangle formula.



Example 2: Approximate $\int_0^8 (3x - 1)dx$ by using 4 rectangles of equal width whose heights are A) the right endpoints, B) the left endpoints, C) the midpoints of each subinterval, and D) by using geometry.

Section 7.4: The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus: Let f be a continuous function on $[a, b]$ and F be any antiderivative of f . Then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Properties of Definite Integrals:

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b kf(x)dx = k \int_a^b f(x)dx$, where k is a constant.
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any number c between a and b
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Example 1: $\int_0^1 2x dx$

$$= 2 \cdot \frac{1}{2} x^2 \Big|_0^1 \quad (\text{no } +C \text{ needed})$$

$$= x^2 \Big|_0^1$$

$$= [(1)^2] - [(0)^2]$$

$$= 1 - 0$$

$$= 1$$

On the Calculator:

First, press the MATH button. Scroll down till you find fnInt(and hit ENTER. Now, enter the function you are integrating; here, I enter 2x. Then enter a comma followed by another x, since we are integrating dx. Finally, enter another comma followed by the lower bound, 0, and a third comma followed by the higher bound, 1. Then hit ENTER to get your answer.



Example 2: $\int_2^5 (6x^2 - 3x + 5) dx$

Example 3: $\int_0^5 x\sqrt{25 - x^2} dx$

Example 4: Find the area of the region between the x-axis and the graph of $f(x) = x^2 - 3x$ from $x = 1$ to $x = 3$.

Section 7.5: Area Between Two Curves

Area between 2 curves: If f and g are continuous functions and $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between f and g from $x = a$ to $x = b$ is given by:

$$A = \int_a^b [f(x) - g(x)] dx$$

Example 1: Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$.

Example 2: Find the area between $y = x^{1/2}$ and $y = x^3$

Example 3: Find the area between $y = 2x$ and $y = x^3$