

# Math 104 – Trigonometry Notes Fall 2013

Intended for use with:

Trigonometry, Tenth Edition

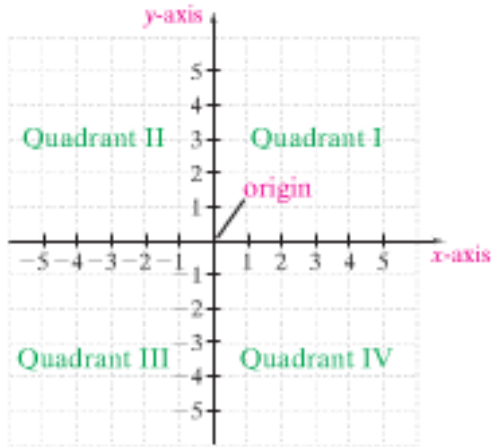
Lial, Hornsby, Schneider

copyright 2013, by Addison-Wesley Educational Publishers, Inc.

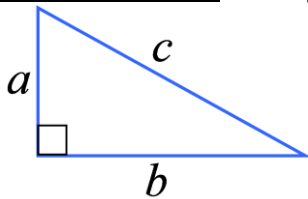
ISBN 10-0-321-67177-5

## Appendix B: Graphs of Equations

### Rectangular Coordinate System:



Pythagorean Theorem: In a right triangle,  $a^2 + b^2 = c^2$



Example 1: Find the third side of the right triangle if  $a = 5$  and  $b = 13$ .

Distance Formula: The distance between the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2: Find the distance between  $(-6, -10)$  and  $(6, 5)$ .

Midpoint Formula: The point halfway between  $(x_1, y_1)$  and  $(x_2, y_2)$  is:

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Example 3: Find the midpoint between (8, -4) and (-9, 6).

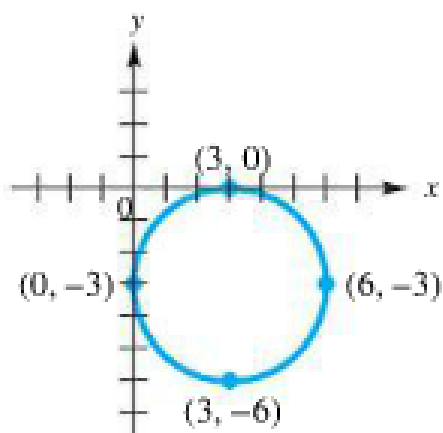
Example 4: Graph:  $y = 4x - 1$

Example 5: Graph:  $y = x^2 - 4$

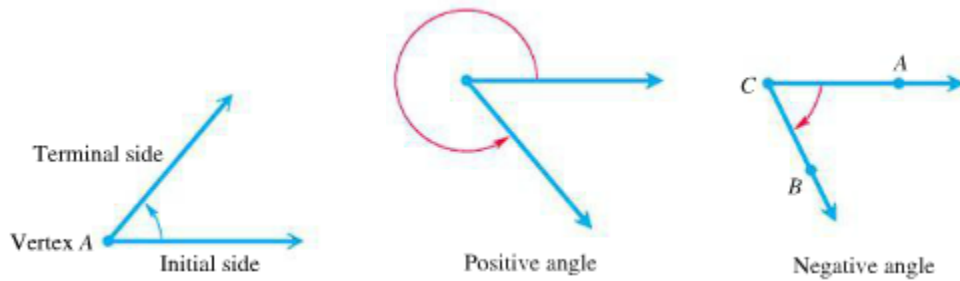
Center-Radius Form of the Equation of a Circle:  $(x - h)^2 + (y - k)^2 = r^2$ , where (h, k) is the center of the circle and r is the radius.

Example 6: Graph:  $(x - 1)^2 + (y + 2)^2 = 9$

Example 7: Find the equation of the circle:



## 1.1: Angles



Circle, 1 Rotation:  $360^\circ$

Acute Angle: under  $90^\circ$

Obtuse Angle: between  $90^\circ$  and  $180^\circ$

Right Angle:  $90^\circ$

Straight Angle:  $180^\circ$

Complimentary Angles: 2 angles whose sum is  $90^\circ$

Supplementary Angles: 2 angles whose sum is  $180^\circ$

Example 1: Find the angle complimentary to and supplementary to  $40^\circ$ .

Measure:  $m(\text{angle } A)$

Minute:  $1' = 1/60$  degrees

Second:  $1'' = 1/60$  minutes =  $1/3600$  degrees

Example 2: Add:  $51^\circ 29' + 32^\circ 46'$

Standard Position: angle drawn with one side along positive x-axis, measure in the positive direction.

Quadrantal Angles: An angle in standard position whose terminal side lies on an axis:  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $360^\circ$ .

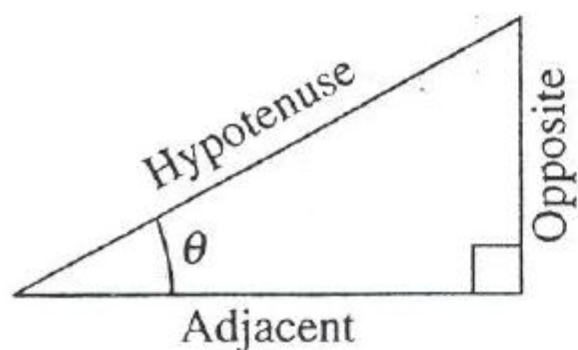
Coterminal Angles: angles whose measures differ by a multiple of  $360^\circ$ .

Example 3: Find the angle with the least possible positive measure that is coterminal:

A)  $908^\circ$

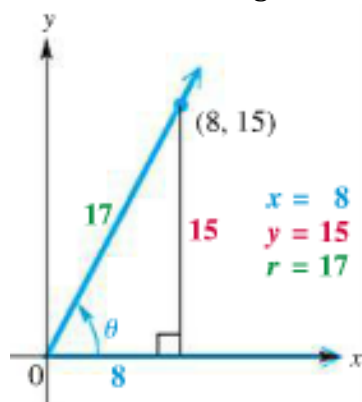
B)  $-75^\circ$

### 1.3: Trigonometric Functions

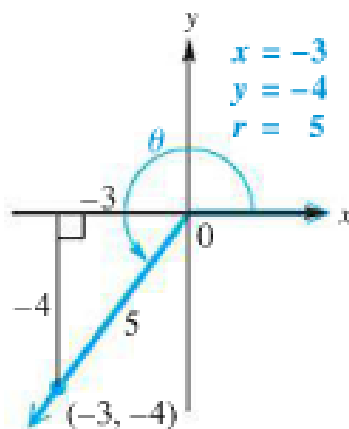


$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}}\end{aligned}$$

Example 1: Find all six trigonometric functions for the given angle:



A)



B)

Function Values of Quadrantal Angles:

|                       | Sin | Cos | Tan   | Cot   | Sec   | Csc   |
|-----------------------|-----|-----|-------|-------|-------|-------|
| $0^\circ / 360^\circ$ | 0   | 1   | 0     | Undef | 1     | Undef |
| $90^\circ$            | 1   | 0   | Undef | 0     | Undef | 1     |
| $180^\circ$           | 0   | -1  | 0     | Undef | -1    | Undef |
| $270^\circ$           | -1  | 0   | Undef | 0     | Undef | -1    |

Example 2: Find the following:

A)  $3\cos 0^\circ - 4 \sec 180^\circ$

B)  $\sin^2 270^\circ + \cot 90^\circ \csc 90^\circ$

1.4: Using the Definitions of the Trigonometric FunctionsReciprocal Identities:

$$\sin x = \frac{1}{\csc x} \quad \sec x = \frac{1}{\cos x} \quad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \quad \cos x = \frac{1}{\sec x} \quad \cot x = \frac{1}{\tan x}$$

Example 1: Find  $\cos \theta$  given  $\sec \theta = 5/3$ Example 2: Find  $\sin \theta$  given  $\csc \theta = -\frac{\sqrt{12}}{2}$

Signs of Functions:

|     | Sin | Cos | Tan | Cot | Sec | Csc |
|-----|-----|-----|-----|-----|-----|-----|
| I   | +   | +   | +   | +   | +   | +   |
| II  | +   | -   | -   | -   | -   | +   |
| III | -   | -   | +   | +   | -   | -   |
| IV  | -   | +   | -   | -   | +   | -   |

Quadrant II: Sin, Csc positive

Quadrant I: All positive

Quadrant III: Tan, Cot positive

Quadrant IV: Cos, Sec positive

Example 3: Determine the signs of the six trigonometric functions for each angle:A)  $87^\circ$ B)  $300^\circ$ C)  $-200^\circ$ Example 4: Identify the quadrant(s) for  $\theta$  that satisfy the given conditions:A)  $\sin \theta > 0, \tan \theta < 0$ B)  $\cos \theta < 0, \sec \theta < 0$

### Ranges of Trigonometric Functions

| Trigonometric Function of $\theta$ | Range (Set-Builder Notation)            | Range (Interval Notation)        |
|------------------------------------|-----------------------------------------|----------------------------------|
| $\sin \theta, \cos \theta$         | $\{y \mid  y  \leq 1\}$                 | $[-1, 1]$                        |
| $\tan \theta, \cot \theta$         | $\{y \mid y \text{ is a real number}\}$ | $(-\infty, \infty)$              |
| $\sec \theta, \csc \theta$         | $\{y \mid  y  \geq 1\}$                 | $(-\infty, -1] \cup [1, \infty)$ |

Example 5: Determine whether the statement is possible or impossible:

A)  $\sin \theta = 2.5$

B)  $\tan \theta = 110.47$

C)  $\sec \theta = 0.6$

Example 6: If  $\theta$  is in quadrant II and  $\sin \theta = 2/3$ . Find the values of the other five trigonometric functions.

### Pythagorean Identities

For all angles  $\theta$  for which the function values are defined, the following identities hold.

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

### Quotient Identities

For all angles  $\theta$  for which the denominators are not zero, the following identities hold.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \quad \frac{\cos \theta}{\sin \theta} = \cot \theta$$

Example 7: Find  $\sin \theta$  and  $\tan \theta$  given that  $\cos \theta = -\frac{\sqrt{3}}{4}$  and  $\sin \theta > 0$

Example 8: Find  $\cos \theta$  and  $\sin \theta$  given that  $\tan \theta = 4/3$  and  $\theta$  is in quadrant III.

## 2.1: Trigonometric Functions of Acute Angles

### **Cofunction Identities**

For any acute angle  $A$ , cofunction values of complementary angles are equal.

$$\sin A = \cos(90^\circ - A) \quad \sec A = \csc(90^\circ - A) \quad \tan A = \cot(90^\circ - A)$$

$$\cos A = \sin(90^\circ - A) \quad \csc A = \sec(90^\circ - A) \quad \cot A = \tan(90^\circ - A)$$

Example 1: Write each function in terms of its cofunction:

A)  $\cos 52^\circ$

B)  $\tan 71^\circ$

C)  $\sec 24^\circ$

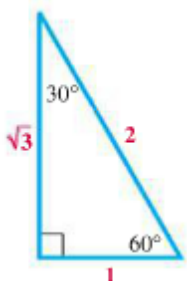
Example 2: Solve using the cofunction identities:

A)  $\cos(\theta + 4^\circ) = \sin(3\theta + 2^\circ)$

B)  $\tan(2\theta - 18^\circ) = \cot(\theta + 18^\circ)$

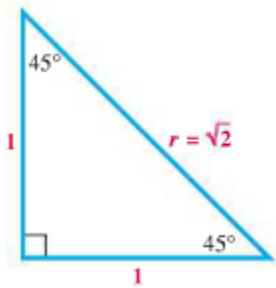
Trigonometric Function Values of Special Angles:

30-60-90 Triangle:



Example 3: Find all 6 trigonometric functions for  $30^\circ$  and  $60^\circ$ .

45-45-90 Triangle:

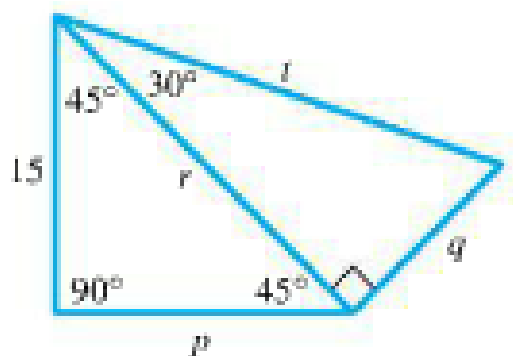


Example 4: Find all 6 trigonometric functions for  $45^\circ$ .

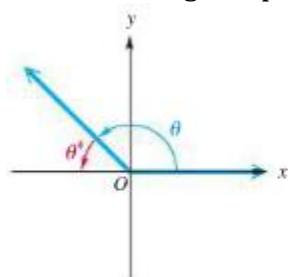
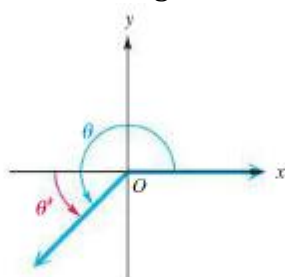
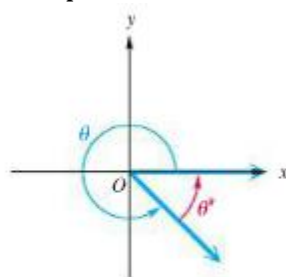
Function Values of Special Angles:

|            | Sin          | Cos          | Tan          | Cot          | Sec           | Csc           |
|------------|--------------|--------------|--------------|--------------|---------------|---------------|
| $30^\circ$ | $1/2$        | $\sqrt{3}/2$ | $\sqrt{3}/3$ | $\sqrt{3}$   | $2\sqrt{3}/3$ | $2$           |
| $45^\circ$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | $1$          | $1$          | $\sqrt{2}$    | $\sqrt{2}$    |
| $60^\circ$ | $\sqrt{3}/2$ | $1/2$        | $\sqrt{3}$   | $\sqrt{3}/3$ | $2$           | $2\sqrt{3}/3$ |

Example 5: Find the remaining sides of the triangle:

2.2: Trigonometric Functions of Non-Acute Angles

Reference Angle: a positive acute angle found with respect to the x-axis.

 $\theta$  in quadrant II $\theta$  in quadrant III $\theta$  in quadrant IV

Example 1: Find the reference angle:

A)  $218^\circ$

B)  $1387^\circ$

C)  $488^\circ$

Example 2: Find all six trigonometric functions for  $210^\circ$ .

Step 1: If angle  $> 360$ , find the coterminal angle.

Step 2: Find the reference angle.

Step 3: Find trig values.

Step 4: Find signs

Example 3: Find all six trigonometric functions for  $135^\circ$ .

Example 4: Find the exact value:

A)  $\cos (-240^\circ)$

B)  $\tan (675^\circ)$

Example 5: Evaluate:  $\cos 120^\circ + 2\sin^2 60^\circ - \tan^2 30^\circ$

Example 6: Evaluate the expression:

A)  $\cos 780^\circ$

B)  $\cot (-405^\circ)$

Example 7: Find  $\theta$  given that  $\cos \theta = -\frac{\sqrt{2}}{2}$  and the interval  $[0^\circ, 360^\circ)$

### 2.3: Finding Trigonometric Function Values Using a Calculator

For all problems, check the MODE. Here, we want degree mode.

Example 1: Use the calculator to approximate. Round to the nearest thousandth.

A)  $\sin 49^\circ 12'$

B)  $\sec 97.977^\circ$

C)  $\frac{1}{\cot 51.4283^\circ}$

D)  $\sin 30^\circ$

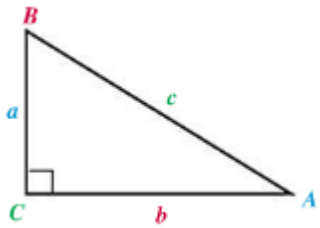
Example 2: Use the calculator to find  $\theta$  over  $[0^\circ, 90^\circ]$  that satisfies the given condition:

A)  $\sin \theta = 0.9677091705$

B)  $\cos \theta = 0.921$

C)  $\sec \theta = 1.054$

#### 2.4: Solving Right Triangles



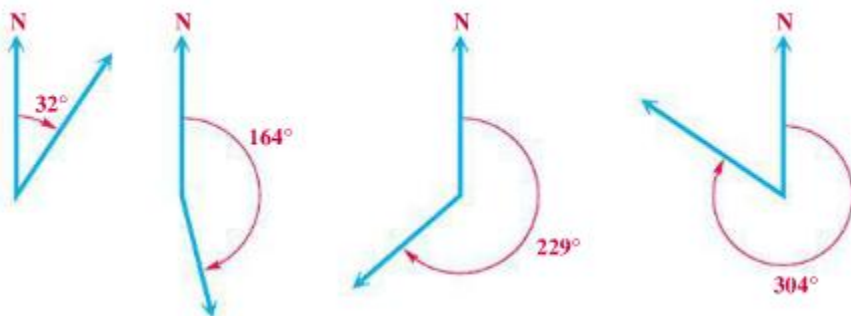
Example 1: Solve the right triangle ABC if  $A = 34^\circ 30'$  and  $c = 12.7$  in.

Example 2: Solve right triangle ABC if  $a = 29.43$  cm and  $c = 53.58$  cm.

Example 3: Sally stands 123 feet from a flagpole. The angle of elevation to the top of the flagpole is  $26^\circ 40'$ . Sally's eyes are 5.30 feet above the ground. Find the height of the flagpole.

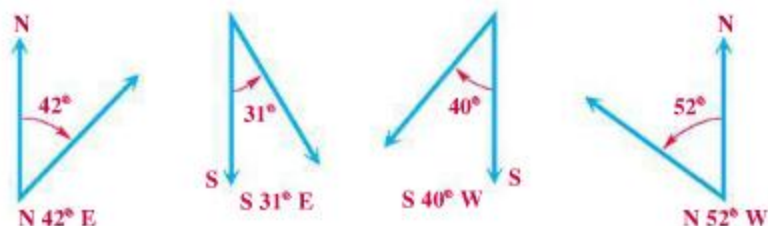
## 2.5: Further Applications of Right Triangles

### Bearing, method 1:



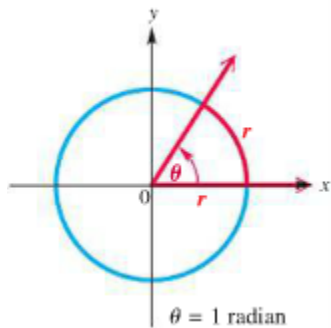
**Example 1:** Radar stations A and B are on an east-west line, 3.7 kilometers apart. Station A detects a plane at point C on a bearing of  $61^\circ$ . At the same time, station B detects the same plane on a bearing of  $331^\circ$ . Find the distance from station A to point C.

### Bearing, method 2:



**Example 2:** A ship leaves port and sails on a bearing of  $N 47^\circ E$  for 3.5 hours. It then turns and sails on a bearing of  $S 43^\circ E$  for 4.0 hours. If the ship's rate of speed is 22 knots, find the distances that the ship is from port.

### 3.1: Radian Measure



Radian: an angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of 1 radian.

If no unit is specified, assumed the angle is in radians. Usually an angle measured in radians includes  $\pi$ .

$$180^\circ = \pi \text{ radians}$$

$$360^\circ = 2\pi \text{ radians}$$

Example 1: Convert the angle to radians:

A)  $45^\circ$

To convert to radians, multiply by  $\frac{\pi}{180^\circ}$

B)  $-270^\circ$

C)  $249.8^\circ$  (round to two decimals)

Example 2: Convert the angle to degrees:

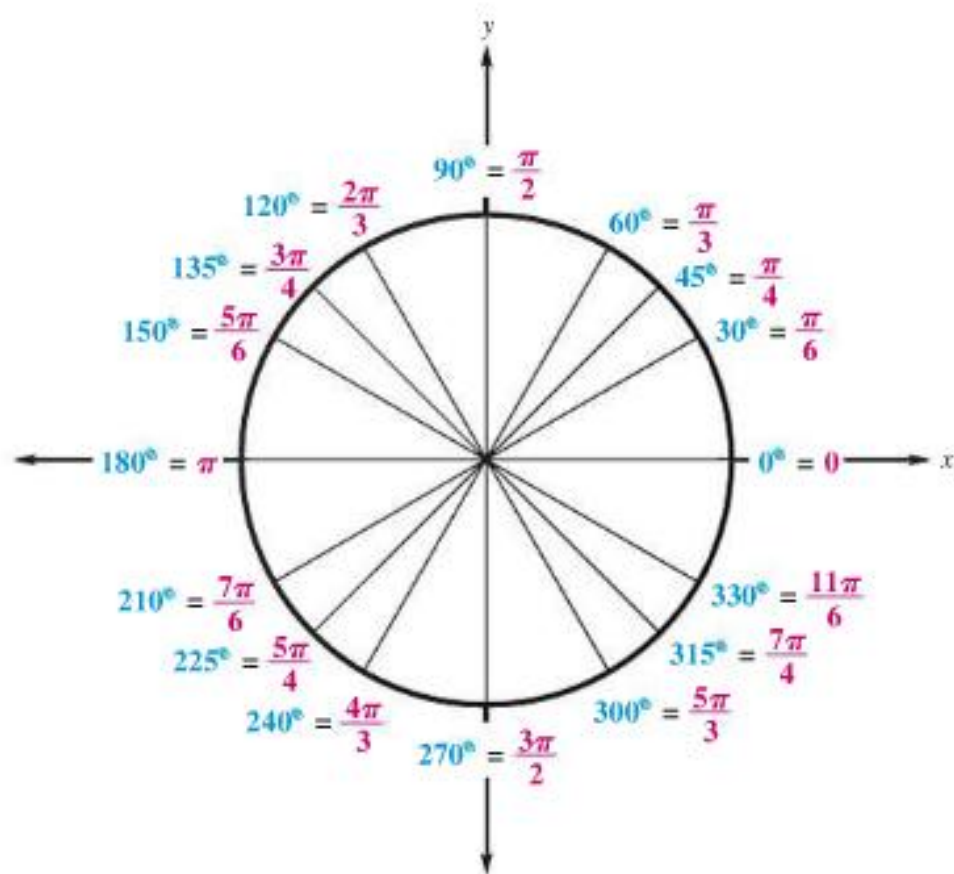
A)  $\frac{9\pi}{4}$

To convert, multiply by  $\frac{180^\circ}{\pi}$

B)  $\frac{-5\pi}{6}$

C)  $4.25$  (round to two decimals)

Some Angles in Degrees and Radians:



Example 3: Find the exact value:

A)  $\tan \frac{2\pi}{3}$

B)  $\sin \frac{3\pi}{2}$

C)  $\cos \frac{-4\pi}{3}$

### 3.2: Applications of Radian Measure

Arc Length: The length  $s$  of an arc intercepted on a circle of radius  $r$  by a central angle  $\theta$  radians is  $s = r\theta$ .

Example 1: A circle has a radius of 18.20 cm. Find the arc length for the following angles:

A)  $\theta = \frac{3\pi}{8}$

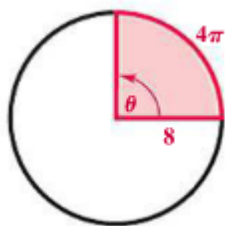
B)  $\theta = 144^\circ$

Example 2: The tip of the minute hand of a clock is 2 in from the center. For 20 min, how far has the tip of the minute hand traveled?

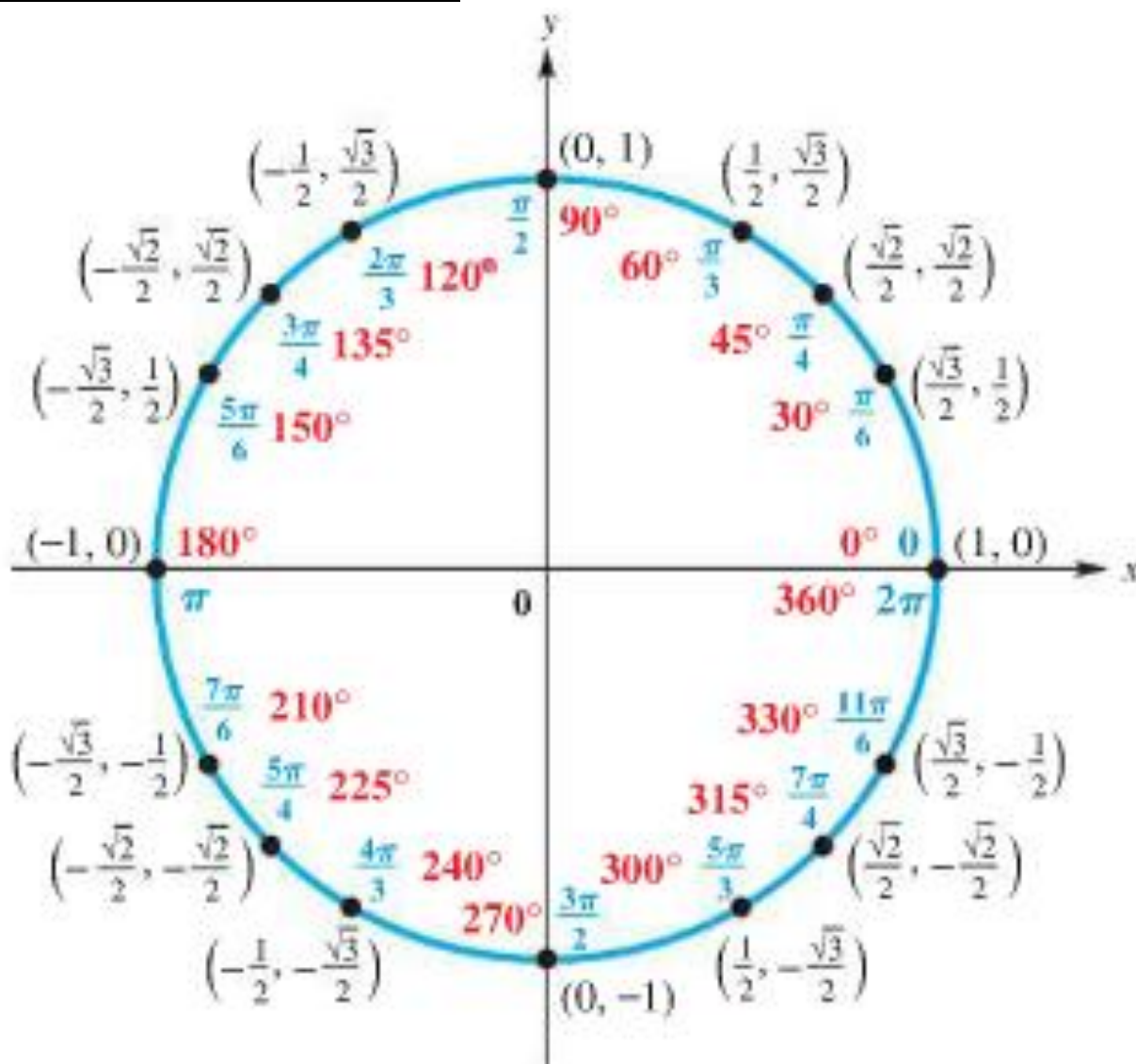
Example 3: The latitude of Reno, NV is  $40^\circ$  N which is approximately due north of LA, whose latitude is  $34^\circ$  N. The radius of the Earth is 6400 km. Find the distance between the cities.

Area of a Sector: The area of a sector with radius  $r$  and central angle  $\theta$  is  $A = \frac{1}{2}r^2\theta$

Example 3: Find the area of the sector:



### 3.3: The Unit Circle and Circular Functions



#### Circular Functions

For any real number  $s$  represented by a directed arc on the unit circle,

$$\begin{array}{lll} \sin s = y & \cos s = x & \tan s = \frac{y}{x} \quad (x \neq 0) \\ \csc s = \frac{1}{y} \quad (y \neq 0) & \sec s = \frac{1}{x} \quad (x \neq 0) & \cot s = \frac{x}{y} \quad (y \neq 0). \end{array}$$

#### Domains:

Sin, Cos:  $(-\infty, \infty)$

Tan, Sec: {all  $s$  such that  $s \neq (2n + 1)\frac{\pi}{2}$  where  $n$  is an integer}

Not  $\pi/2, -\pi/2, 3\pi/2$ , etc.

Cot, Csc: {all  $s$  such that  $s \neq n\pi$  where  $n$  is an integer}

Not  $0, \pi, 2\pi$ , etc.

Example 1: Find the exact value:

A)  $\sin \frac{3\pi}{2}$

B)  $\tan \frac{3\pi}{2}$

C)  $\cos \frac{7\pi}{4}$

D)  $\tan \left( \frac{-5\pi}{3} \right)$

Example 2: Use a calculator to approximate:

A)  $\cos 1.85$

B)  $\cos 0.5149$

C)  $\cot 1.3209$

Example 3: Find the value of s using the given interval:

A)  $\cos s = -1/2$  over  $[\pi/2, \pi]$

B)  $\tan^2 s = 3$  over  $[0, 2\pi)$

### 3.4: Linear and Angular Speed

Angular Speed:  $\omega = \frac{\theta}{t}$  where  $\theta$  is in radians.

Linear Speed:  $v = r\omega = \frac{s}{t}$  where  $s$  is arc length

Example 1: The radius of a circle is 10 cm and the angular speed is  $\pi/18$  rad/sec.

A) Find the angle after 6 seconds.

B) Find the distance traveled in 6 seconds.

C) Find the linear speed.

Example 2: A belt runs a pulley of radius 6 cm at 80 revolutions per min.

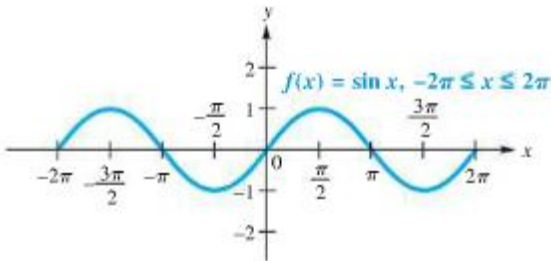
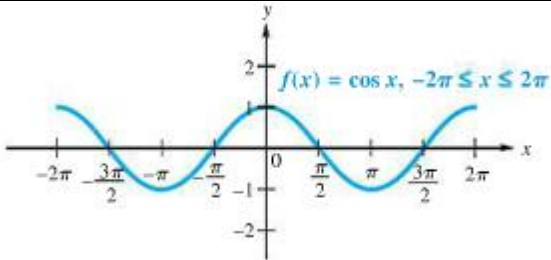
A) Find the angular speed of the pulley in radians per second.

B) Find the linear speed of the belt in cm/sec.

#### 4.1: Graphs of the Sine and Cosine Functions

**Periodic Function:** a function such that  $f(x) = f(x + np)$ , for every real number  $x$ , every integer  $n$ , and some positive real number  $p$ .

The least possible value of  $p$  is the period of the function.

| Function                                                                                             | Graph                                                                              | Domain and Range                       |
|------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|----------------------------------------|
| Sine: $f(x) = \sin x$<br><br>Continuous<br>Period: $2\pi$<br>x-intercepts: $n\pi$                    |  | D: $(-\infty, \infty)$<br>R: $[-1, 1]$ |
| Cosine: $f(x) = \cos x$<br><br>Continuous<br>Period: $2\pi$<br>x-intercepts: $(2n + 1)\frac{\pi}{2}$ |  | D: $(-\infty, \infty)$<br>R: $[-1, 1]$ |

Standard Trigonometry Viewing Window:

Xmin = 0, xmax =  $2\pi$ , xscl =  $\pi/2$ , ymin = -2, ymax = 2, yscl = 1

Amplitude: the graph of  $y = a \sin x$  or  $y = a \cos x$  ( $a \neq 0$ ) will have a range of  $[-|a|, |a|]$

Example 1: Find the amplitude and graph:

A)  $y = 2 \sin x$

B)  $y = \frac{1}{3} \cos x$

Period: For  $b > 0$ , the graph of  $y = \sin bx$  or  $y = \cos bx$  will have a period of  $\frac{2\pi}{b}$

Hash mark: List 5 points on every graph: the beginning and ending of one period, the  $1/4$  mark, the halfway point, and the  $3/4$  mark. To find the hash marks, add  $\pi/2b$  to the start point.

Example 2: Find the period and graph:

A)  $y = \sin 2x$

B)  $y = \cos \frac{2}{3}x$

Example 3: Graph over one period:  $y = -2 \sin 3x$

Example 4: Graph over one period:  $y = 3 \cos \pi x$

#### 4.2: Translations of the Graphs of Sine and Cosine

Horizontal Translation: the graph of  $y = \sin (x - d)$  or  $y = \cos (x - d)$  is shifted  $d$  units to the *right*.

Example 1: Graph over one period:

A)  $y = \cos(x - \frac{\pi}{3})$

B)  $y = 3 \sin(x + \frac{\pi}{4})$

C)  $y = -2 \cos (3x + \pi)$

Vertical Translation: the graph of  $y = c + f(x)$  is shifted  $c$  units up.

Example 2: Graph over 2 periods:

A)  $y = -1 + \sin x$

B)  $y = 3 - \cos 3x$

Summary of Transformations: The graph of  $y = c + a \sin b(x - d)$  or  $y = c + a \cos b(x - d)$  has:

Amplitude:  $|a|$

Period:  $2\pi/b$

Shifted up  $c$  units

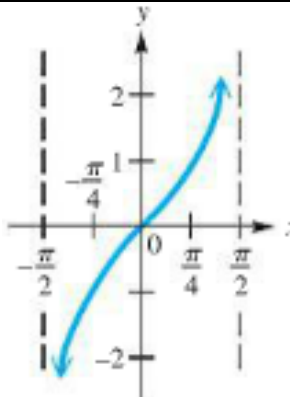
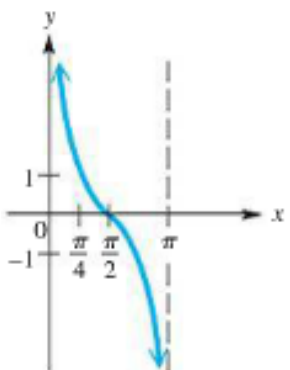
Shifted right  $d$  units

Is flipped upside down if  $a < 0$

Hash marks: add  $\pi/2b$  to the start point.

Example 3: Graph:  $y = -1 + 2 \sin (4x + \pi)$

#### 4.3: The Graphs of Tangent and Cotangent

| Function                                                                                                                                                                                               | Graph                                                                               | Domain and Range                                                                                                               |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|
| <p>Tangent: <math>f(x) = \tan x</math></p> <p>Discontinuous at vertical asymptotes every <math>(2n + 1) \frac{\pi}{2}</math></p> <p>Period: <math>\pi</math><br/>x-intercepts: <math>n\pi</math></p>   |  | <p>D: <math>\{x \neq (2n + 1) \frac{\pi}{2}\}</math><br/>R: <math>(-\infty, \infty)</math></p>                                 |
| <p>Cotangent: <math>f(x) = \cot x</math></p> <p>Discontinuous at vertical asymptotes every <math>n\pi</math></p> <p>Period: <math>\pi</math><br/>x-intercepts: <math>(2n + 1) \frac{\pi}{2}</math></p> |  | <p>D: <math>\{x \neq n\pi\}</math><br/>R: <math>(-\infty, \infty)</math></p> <p>To graph, enter as <math>1 / \tan x</math></p> |

Summary of Transformations: The graph of  $y = c + a \tan b(x - d)$  or  $y = c + a \cot b(x - d)$  has:

Period:  $\pi/b$

Shifted up  $c$  units

Shifted right  $d$  units

“Amplitude”:  $|a|$  (measure of how far it’s stretched; not really amplitude)

Is flipped upside down if  $a < 0$

Hash marks: add  $\pi/2b$  to the start point.

Example 1: Graph:  $y = \tan 2x$

Example 2: Graph:  $y = -3 \tan \frac{1}{2}x$

Example 3: Graph:  $y = \frac{1}{2} \cot 2x$

Example 4: Graph:  $y = 2 + \tan \pi x$

Example 5: Graph:  $y = -2 - \cot(x - \frac{\pi}{4})$

### 5.1: Fundamental Identities

#### **Fundamental Identities**

##### **Reciprocal Identities**

$$\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}$$

##### **Quotient Identities**

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

##### **Pythagorean Identities**

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

##### **Negative-Angle Identities**

$$\begin{array}{lll} \sin(-\theta) = -\sin \theta & \cos(-\theta) = \cos \theta & \tan(-\theta) = -\tan \theta \\ \csc(-\theta) = -\csc \theta & \sec(-\theta) = \sec \theta & \cot(-\theta) = -\cot \theta \end{array}$$

Example 1: If  $\tan \theta = \frac{-5}{3}$  and  $\theta$  is in Quadrant II, find:

A)  $\sec \theta$

B)  $\sin \theta$

C)  $\cot (-\theta)$

Example 2: Write  $\cos x$  in terms of  $\tan x$

Example 3: Write  $\tan x + \cot x$  in terms of  $\sin x$  and  $\cos x$ . Simplify.

Example 4: Write  $\frac{1 + \cot^2 \theta}{1 - \csc^2 \theta}$  in terms of  $\sin \theta$  and  $\cos \theta$ . Then simplify the expression so that no quotients appear.

## 5.2: Verifying Trigonometric Identities

### Hints:

Work the more complicated side.

Write terms as  $\sin$  and  $\cos$

Try factoring or other algebraic techniques

Keep end property in mind

Example 1: Verify:  $\cot \theta + 1 = \csc \theta (\cos \theta + \sin \theta)$

Example 2: Verify:  $\tan^2 x (1 + \cot^2 x) = \frac{1}{1 - \sin^2 x}$

Example 3: Verify:  $\frac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

Example 4: Verify:  $\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$

Example 5: Verify:  $\frac{\sec \theta + \tan \theta}{\sec \theta - \tan \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{\cos^2 \theta}$

### 5.3: Sum and Difference Identities for Cosine

#### Cosine of a Sum or Difference

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Example 1: Find the exact value:

A)  $\cos 15^\circ$

B)  $\cos \frac{5\pi}{12}$

C)  $\cos 87^\circ \sin 93^\circ - \sin 87^\circ \cos 93^\circ$

#### Cofunction Identities

The following identities hold for any angle  $\theta$  for which the functions are defined.

$$\cos(90^\circ - \theta) = \sin \theta \quad \cot(90^\circ - \theta) = \tan \theta$$

$$\sin(90^\circ - \theta) = \cos \theta \quad \sec(90^\circ - \theta) = \csc \theta$$

$$\tan(90^\circ - \theta) = \cot \theta \quad \csc(90^\circ - \theta) = \sec \theta$$

Example 2: Find  $\theta$ :

A)  $\cot \theta = \tan 25^\circ$

B)  $\sin \theta = \cos (-30^\circ)$

C)  $\csc \frac{3\pi}{4} = \sec \theta$

Example 3: Write  $\cos (180^\circ - \theta)$  as a trigonometric function of  $\theta$  alone.

Example 4: Suppose  $\sin s = 3/5$ ,  $\cos t = -12/13$ , and both  $s$  and  $t$  are in quadrant II. Find  $\cos (s + t)$

### 5.4: Sum and Difference Identities for Sine and Tangent

#### Sine of a Sum or Difference

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

#### Tangent of a Sum or Difference

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 1: Find the exact value:

A)  $\sin 75^\circ$

B)  $\tan \frac{7\pi}{12}$

C)  $\sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ$

Example 2: Write as an expression involving  $\theta$ :

A)  $\sin (30^\circ + \theta)$

B)  $\tan (45^\circ - \theta)$

C)  $\sin (180^\circ - \theta)$

Example 3: Suppose  $\sin A = 4/5$ , with  $\pi/2 < A < \pi$ , and  $\cos B = -5/13$ , with  $\pi < B < 3\pi/2$ . Find:

A)  $\sin (A + B)$

B)  $\tan (A + B)$

C) the quadrant of  $(A + B)$

### 5.5: Double Angle Identities

#### Double-Angle Identities

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \cos 2A = 1 - 2 \sin^2 A$$

$$\cos 2A = 2 \cos^2 A - 1 \qquad \sin 2A = 2 \sin A \cos A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Example 1: Given  $\cos \theta = 3/5$  and  $\sin \theta < 0$ , find  $\sin 2\theta$ ,  $\cos 2\theta$ , and  $\tan 2\theta$

Example 2: Find the values of the six trigonometric functions of  $\theta$  if  $\cos 2\theta = 4/5$  and  $90^\circ < \theta < 180^\circ$ .

Example 3: Simplify the expression:

A)  $\cos^2 7x - \sin^2 7x$

B)  $\sin 15^\circ \cos 15^\circ$

Example 4: Write  $\sin 3x$  in terms of  $\sin x$ .

#### Product-to-Sum Identities

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]$$

Example 5: Write  $4 \cos 75^\circ \sin 25^\circ$  as the sum or difference of two functions.

### Sum-to-Product Identities

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Example 6: Write  $\sin 2\theta - \sin 4\theta$  as a product of two functions.

### 5.6: Half-Angle Identities

#### Half-Angle Identities

In the following identities, the symbol  $\pm$  indicates that the sign is chosen based on the function under consideration and the quadrant of  $\frac{A}{2}$ .

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \quad \sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \quad \tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} \quad \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

Example 1: Find the exact value of  $\cos 15^\circ$ .

Example 2: Find the exact value of  $\tan 22.5^\circ$ .

Example 3: Given  $\cos s = 2/3$ , with  $3\pi/2 < s < 2\pi$ , find  $\sin s/2$ ,  $\cos s/2$ , and  $\tan s/2$ .

Example 4: Simplify:

A)  $\pm \sqrt{\frac{1 + \cos 12x}{2}}$

B)  $\frac{1 - \cos 5X}{\sin 5X}$

## 7.1: Oblique Triangles and the Law of Sines

### **Law of Sines**

In any triangle  $ABC$ , with sides  $a$ ,  $b$ , and  $c$ ,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{a}{\sin A} = \frac{c}{\sin C}, \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This can be written in compact form as follows.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 1: Solve triangle  $ABC$  if  $A = 32.0^\circ$ ,  $B = 81.8^\circ$ , and  $a = 42.9$  cm.

Example 2: If  $C = 112.90^\circ$ ,  $A = 31.10^\circ$ , and  $b = 347$  ft, find  $a$ .

### Area of a Triangle (SAS)

In any triangle  $ABC$ , the area  $\mathcal{A}$  is given by the following formulas.

$$\mathcal{A} = \frac{1}{2}bc \sin A, \quad \mathcal{A} = \frac{1}{2}ab \sin C, \quad \text{and} \quad \mathcal{A} = \frac{1}{2}ac \sin B$$

Example 4: Find the area of triangle  $ABC$  if  $B = 55^\circ 10'$ ,  $a = 34.0$  ft, and  $c = 42.0$  ft.

Example 5: Find the area of triangle  $ABC$  if  $A = 24^\circ 40'$ ,  $C = 52^\circ 40'$ , and  $b = 27.3$  cm.

## 7.2: The ambiguous Case of the Law of Sines

### Applying the Law of Sines

1. For any angle  $\theta$  of a triangle,  $0 < \sin \theta \leq 1$ . If  $\sin \theta = 1$ , then  $\theta = 90^\circ$  and the triangle is a right triangle.
2.  $\sin \theta = \sin(180^\circ - \theta)$  (Supplementary angles have the same sine value.)
3. The smallest angle is opposite the shortest side, the largest angle is opposite the longest side, and the middle-valued angle is opposite the intermediate side (assuming the triangle has sides that are all of different lengths).

Example 1: Solve triangle  $ABC$  if  $B = 55^\circ 40'$ ,  $b = 8.94$  m, and  $a = 25.1$  m.

Example 2: Solve triangle ABC if  $A = 55.3^\circ$ ,  $a = 22.8$  ft, and  $b = 24.9$  ft.

Example 3: Solve triangle ABC if  $A = 43.5^\circ$ ,  $a = 10.7$  in, and  $c = 7.2$  in.

### 7.3: The Law of Cosines

Triangle Side Length Restriction: In any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

#### **Law of Cosines**

In any triangle  $ABC$ , with sides  $a$ ,  $b$ , and  $c$ , the following hold.

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

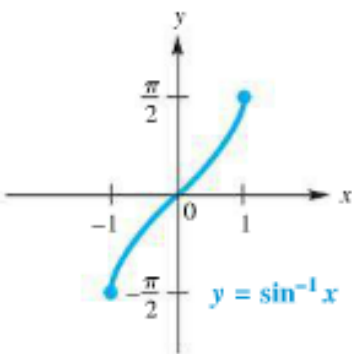
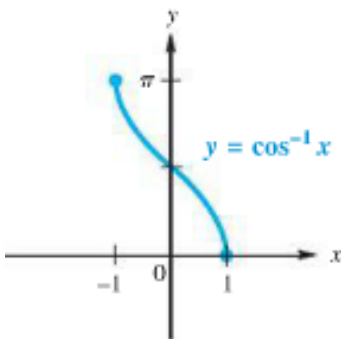
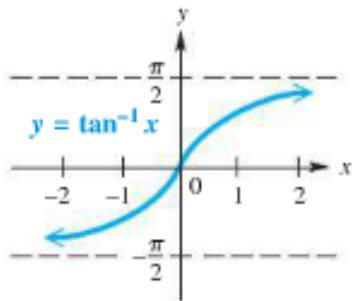
Example 1: If  $b = 259$  m,  $a = 423$  m, and angle  $ACB = 132^\circ 40'$ , find  $c$ .

Example 2: Solve triangle  $ABC$  if  $A = 42.3^\circ$ ,  $b = 12.9$  m, and  $c = 15.4$  m.

Example 3: Solve triangle ABC if  $a = 9.47$  ft,  $b = 15.9$  ft, and  $c = 21.1$  ft.

Example 4: Find angle B if  $a = 11$  ft,  $b = 6$  ft, and  $c = 9$  ft.

## 6.1: Inverse Circular Functions

| Function                                                                                                    | Graph                                                                               | Domain and Range                                                          |
|-------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------|
| Inverse Sine:<br>$y = \sin^{-1} x$<br>$y = \arcsin x$<br>x-intercept, y-intercept: (0, 0).                  |    | D: $[-1, 1]$<br>R: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$           |
| Inverse Cosine:<br>$y = \cos^{-1} x$<br>$y = \arccos x$<br>x-intercept: (0, 1)<br>y-intercept: $(\pi/2, 0)$ |    | D: $[-1, 1]$<br>R: $[0, \pi]$                                             |
| Inverse Tangent:<br>$y = \tan^{-1} x$<br>$y = \arctan x$<br>x-intercept, y-intercept: (0, 0)                |  | D: $(-\infty, \infty)$<br>R: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

Example 1: Find  $y$  in radians.

A)  $y = \arcsin 1/2$

B)  $y = \sin^{-1}(-1)$

C)  $y = \sin^{-1}(-2)$

D)  $y = \arccos 1$

E)  $y = \cos^{-1} \left(-\frac{\sqrt{2}}{2}\right)$

F)  $y = \arctan 1$

G)  $y = \sec^{-1} 2$

H)  $y = \csc^{-1}(-3)$

I)  $y = \operatorname{arccot}(-0.3541)$

Example 2: Evaluate without a calculator:

A)  $\sin\left(\tan^{-1}\frac{3}{2}\right)$

B)  $\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$

Example 3: Evaluate without a calculator:

A)  $\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right)$

B)  $\tan\left(2\arcsin\frac{2}{5}\right)$

Example 4: Write the trigonometric expression as algebraic expression in  $u$ :

A)  $\sin(\tan^{-1} u)$

B)  $\cos(2 \sin^{-1} u)$

## 6.2: Trigonometric Equations I

Example 1: Solve over  $[0^\circ, 360^\circ)$ :  $2 \sin \theta + 1 = 0$

Example 2: Solve over  $[0^\circ, 360^\circ)$ :  $\sin \theta \tan \theta = \sin \theta$

Example 3: Solve over  $[0, 2\pi)$ :  $2 \tan \theta \sin \theta - \tan \theta = 0$

Example 4: Solve over  $[0, 2\pi)$ :  $\tan^2 x + \tan x - 2 = 0$

Example 5: Solve over  $[0, 2\pi)$ :  $\tan \theta + \sqrt{3} = \sec \theta$

### 6.3: Trigonometric Equations II

Example 1: Solve over  $[0, 2\pi)$ :  $2 \sin \frac{x}{2} = 1$

Example 2: Solve:  $2 \sin 2x = 1$

Example 3: Solve:  $\tan 4x = 1$

Example 2: Solve over  $[0, 2\pi)$ :  $\cos 2x = \cos x$

Example 3: Solve over  $[0^\circ, 360^\circ)$ :  $4 \sin \theta \cos \theta = \sqrt{3}$

Example 4: Solve over  $[0, 2\pi)$ :  $\tan 3x + \sec 3x = 2$