

Lecture Notes

Math 102 – College Algebra

Spring 2014

Intended for use with **A Graphical Approach to College Algebra, Fifth Edition**

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Section 1.1: Sets of Real Numbers and the Rectangular Coordinate System

Natural Numbers: $\{1, 2, 3, 4, 5, \dots\}$

Whole Numbers: $\{0, 1, 2, 3, 4, 5, \dots\}$

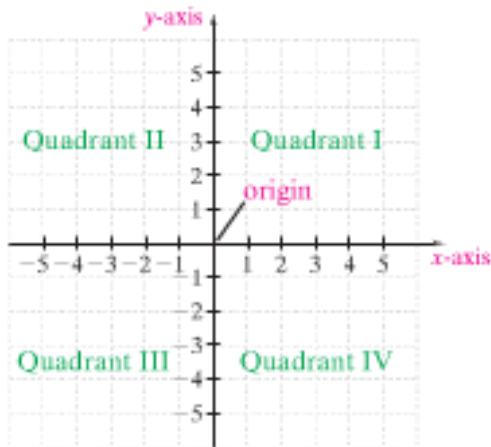
Integers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational Numbers: $\left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers and } b \neq 0 \right\}$
Includes decimals that can be written as fractions.

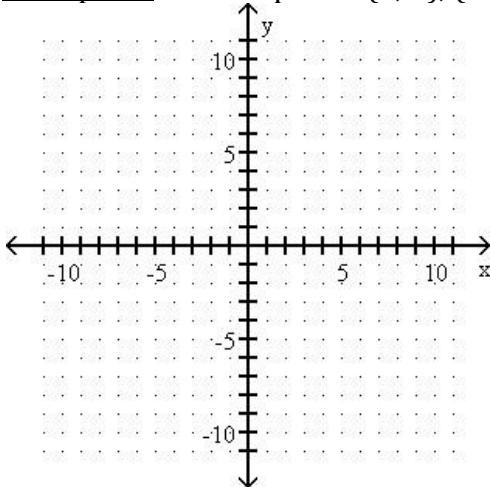
Irrational Numbers: {Numbers that are not rational}
Some irrational numbers: $\sqrt{2}$ or π

Real Numbers: {All numbers that correspond to a point on the number line}
Includes all the rational numbers and all the irrational numbers.

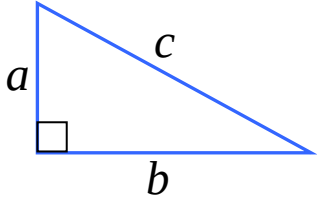
Rectangular Coordinate System:



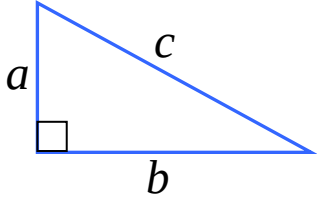
Example 1: Plot the points $(1, 2)$, $(-2, 3)$, $(-3, -4)$, $(4, -5)$, and $(0, 0)$.



Pythagorean Theorem: In a right triangle, $a^2 + b^2 = c^2$



Example 2: Find the third side if $a = 7$ and $b = 24$.



Distance Formula: The distance between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 3: Find the distance between $(-6, -10)$ and $(6, 5)$.

Midpoint Formula: The point halfway between (x_1, y_1) and (x_2, y_2) is: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example 4: Find the midpoint between $(8, -4)$ and $(-9, 6)$.

Section 1.2: Introduction to Relations and Functions

Interval Notation:

Write in the form (Smaller number, larger number) or [smaller number, larger number]

Use () for $<$ and $>$ Don't include the number

Use [] for $\leq$ and $\geq$ Do include the number

Use () for $\pm \infty$ (plus or minus infinity)

Example 1: Write the inequality in interval notation:

A) $x > 3$

B) $x \leq -1$

C) $4 < x \leq 7$

Relation: A set of ordered pairs.

Function: A relation in which every element in the first set corresponds to exactly one element in the second set.

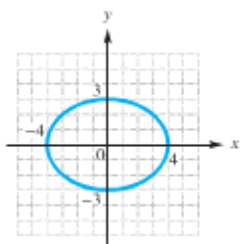
Example 2: Is it a function?

A) $\{(1, 2), (2, 2), (3, 4)\}$

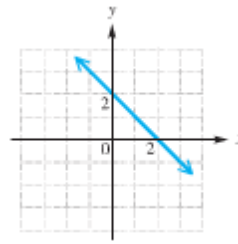
B) $\{(1, 2), (1, 3), (4, 5)\}$

Vertical Line Test: If every vertical line intersects a graph in no more than one point, then the graph is a function.

Example 3: Is it a function?



A)



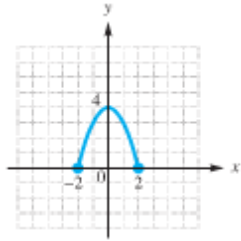
B)

Domain: The x-values in the relation. Cannot divide by zero. Cannot take the square root of a negative.

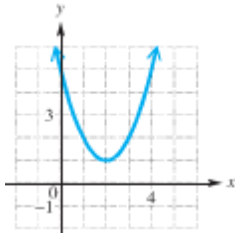
Range: The y-values in the relation.

Example 4: Find the domain and range:

A) $\{(2, 7), (-12, 3), (8, -16)\}$



B)



C)

Function Notation: $f(x) = y$ “the function f of the variable x ” or “ f of x ”

Example 5: Use function notation to evaluate the function $f(x) = x^2 - 3x + 7$.

A) $f(2)$

B) $f(-1)$

C) $f(a + 1)$

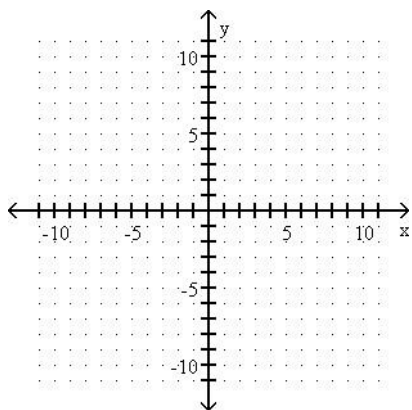
Section 1.3: Linear Functions

Linear Function: $f(x) = ax + b$, where a and b are real numbers.

Example 1: Graph using intercepts: $y = 3x + 6$

x-intercept: where the graph crosses the x-axis. Here, $y = 0$.

y-intercept: where the graph crosses the y-axis. Here, $x = 0$.



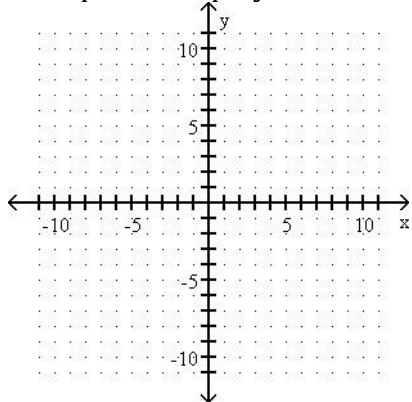
Zero of a Function: Any number c for which $f(c) = 0$. Also called a root, a solution, and an x-intercept.

Example 2: Find the zero of the function $f(x) = 6x - 12$

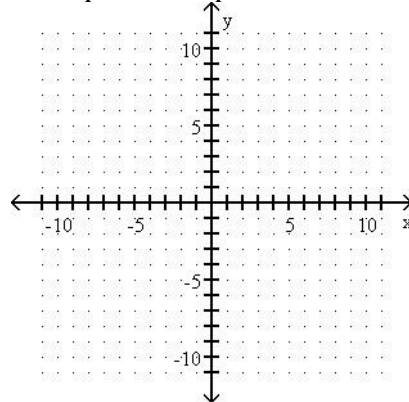
Constant Function: $f(x) = b$ or $y = b$. Makes a horizontal line.

Vertical Line: $x = a$. Not a function.

Example 3: Graph $y = -3$



Example 4: Graph $x = 2$



Slope: The slope of the line through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 5: Find the slope of the line through the given points

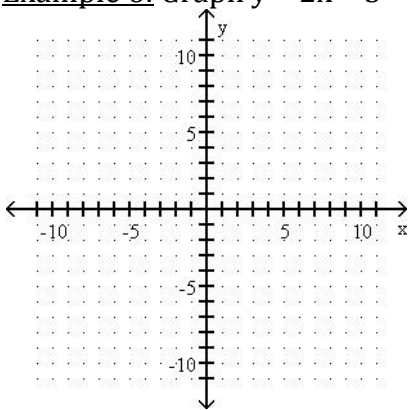
A) $(2, -1)$ and $(-5, 3)$

B) $(0, 4)$ and $(1, 4)$

C) $(4, 0)$ and $(4, 4)$

Slope-Intercept Form of a Line: $y = mx + b$, where m is slope and b is the y -intercept.

Example 6: Graph $y = 2x + 3$



Section 1.4: Equations of Lines

Point-Slope Form of a Line: $y - y_1 = m(x - x_1)$

Standard Form of a Line: $Ax + By = C$, where A , B , and C are integers.

Example 1: Find the equation of the line through $(-1, 3)$ and $(5, -7)$

Example 2: Find the equation of the line through $(-8, 7)$ and $(11, 3)$.

Parallel Lines: have the same slope.

Perpendicular Lines: have negative reciprocal slopes.

$$m_{\perp} = \frac{-1}{m}$$

Example 3: Find the equation of the line parallel to and perpendicular to $3x - 2y = 12$ that passes through $(-3, 4)$.

Section 1.5: Linear Equations and Inequalities

Linear Equation in 1 Variable: of the form $ax + b = 0$.

Example 1: Solve:

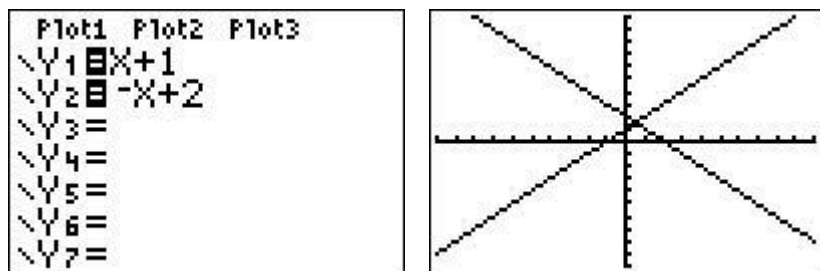
$$10 + 3(2x - 4) = 17 - (x + 5)$$

Example 2: Solve:

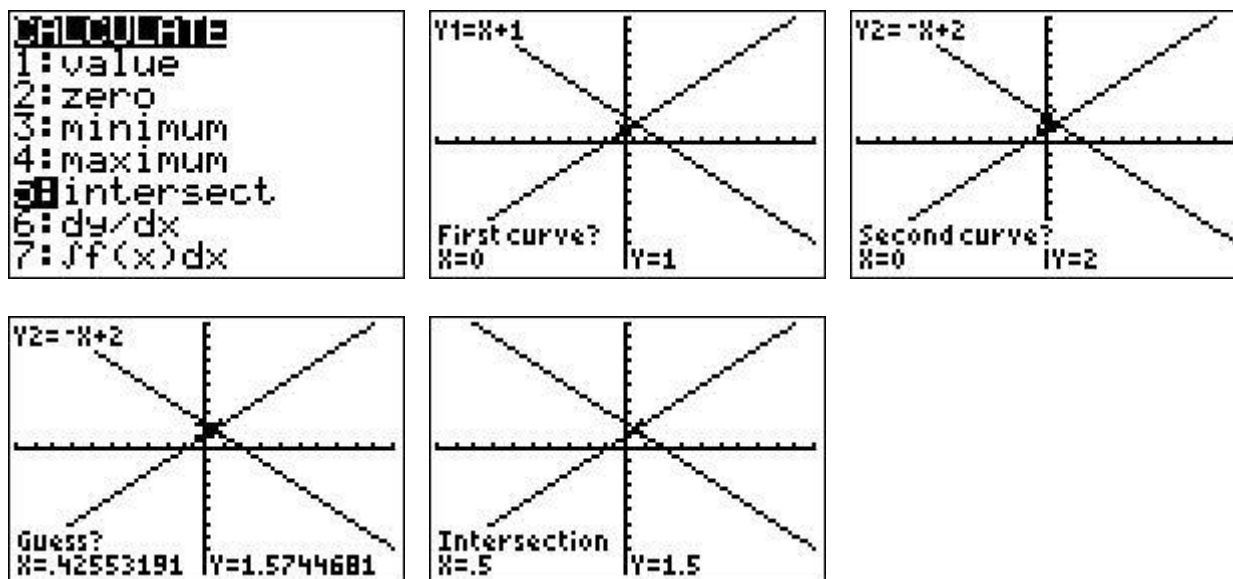
$$\frac{x+7}{6} + \frac{2x-8}{2} = -4$$

Example 3: Solve on the calculator: $x + 1 = -x + 2$

First, enter the two sides of the equation as y1 and y2. The place where the graphs cross is the solution.



For this example, the intersection is not an integer. To find the point where the graphs cross, press 2ND then TRACE for the calc menu. Scroll down till you get to the intersect option and hit ENTER. You will now have to provide some information. First, which 2 curves you want to find the intersection for. Here, I select y1 and then y2. Then, you have to guess where they intersect. Move the blinking cursor until you're close to the place where the lines cross. You don't have to be at the exact point. Hit ENTER and you will be given the point of intersection.



Inequalities:

$<$	"less than"
$>$	"greater than"
$\leq$	"less than or equal to"
$\geq$	"greater than or equal to"
$\neq$	"not equal to"

The inequality sign flips when:

1. Multiplying or dividing by a negative

$$\begin{array}{rcl} -2x & < & 4 \\ x & > & -2 \end{array}$$

2. Rewriting the inequality

$$\begin{array}{rcl} x & > & 3 \\ 3 & < & x \end{array}$$

Example 4: Solve and write your answer in interval notation:

$$3x - 2(2x + 4) \leq 2x + 1$$

Example 5: Solve and write your answer in interval notation:

$$2x - 3 < \frac{x+2}{-3}$$

Example 6: Solve with the calculator and write your answer in interval notation:

$$6 + 3(1 - x) < 0$$

Example 7: Solve and write your answer in interval notation:

$$-2 < 5 + 3x < 20$$

Example 8: Solve and write your answer in interval notation:

$$-10 \geq 3x + 2 \geq -16$$

Section 1.6: Applications of Linear Functions

Example 1: The length of a rectangle is twice its width. If the perimeter is 136 inches, find the dimensions of the rectangle.

Example 2: How much pure alcohol should be mixed with 20 liters of a 40% alcohol solution to get a 50% alcohol solution?

Direct Variation: “y varies directly as x” or “y is directly proportional to x.” Use $y = kx$

Example 3: The distance a spring stretches varies directly as the weight attached to the spring. If a 15 pound weight stretches a spring 8 inches, how far will a 35 pound weight stretch the same spring?

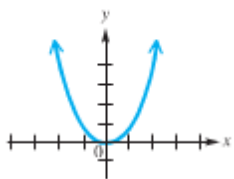
Example 4: Solve the formula $P = 2L + 2W$ for L.

Example 5: Solve the formula $A = \frac{1}{2}h(b_1 + b_2)$ for b_1 .

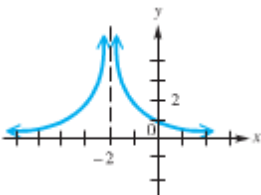
Section 2.1: Graphs of Basic Functions; Symmetry

Continuity (Informal Definition): A function is continuous if you can draw the graph without lifting your pencil from the paper.

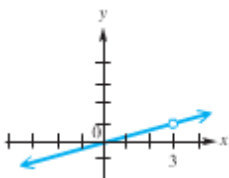
Example 1: Find intervals of continuity:



A)



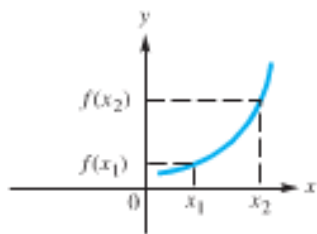
B)



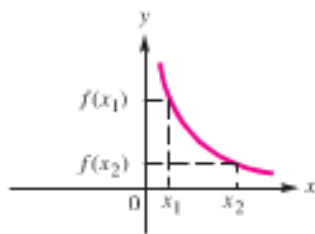
C)

Increasing, Decreasing, and Constant: For a function f defined on an interval I :

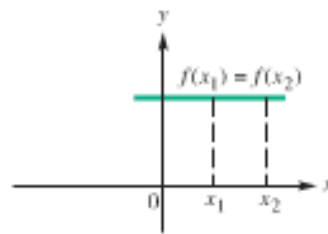
1. f is increasing on I if, when $x_1 < x_2$, $f(x_1) < f(x_2)$
2. f is decreasing on I if, when $x_1 < x_2$, $f(x_1) > f(x_2)$
3. f is constant on I if, for all x_1 and x_2 , $f(x_1) = f(x_2)$.



When $x_1 < x_2$, $f(x_1) < f(x_2)$.
 f is *increasing*.

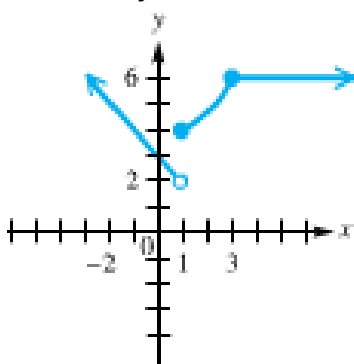


When $x_1 < x_2$, $f(x_1) > f(x_2)$.
 f is *decreasing*.



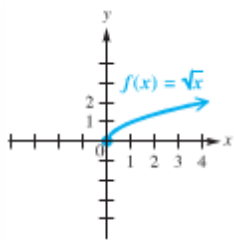
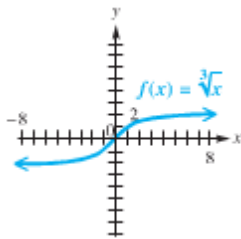
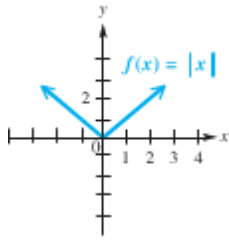
For any x_1 and x_2 , $f(x_1) = f(x_2)$.
 f is *constant*.

Example 2: Identify the intervals over which the function is increasing, decreasing, and constant:

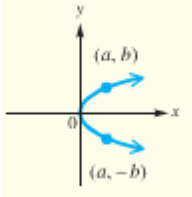
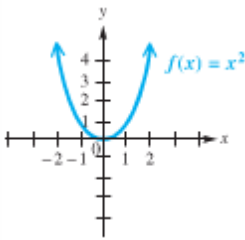
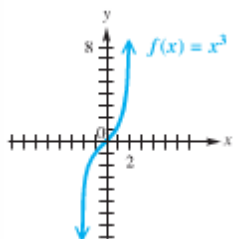


Library of Basic Functions:

Name and Equation	Graph	Domain and Range
Linear Function $f(x) = x$		Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$
Square Function $f(x) = x^2$		Domain: $(-\infty, \infty)$ Range: $[0, \infty)$
Cube Function $f(x) = x^3$		Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

Square Root Function $f(x) = \sqrt{x}$		Domain: $[0, \infty)$ Range: $[0, \infty)$
Cube Root Function $f(x) = \sqrt[3]{x}$		Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$
Absolute Value Function $f(x) = x $		Domain: $(-\infty, \infty)$ Range: $[0, \infty)$

Symmetry:

Type	Example	Graphical Test	Algebraic Test
x-axis		Fold the graph along the x-axis to see if the two sides match up.	If replacing y with $-y$ results in the same equation.
y-axis		Fold the graph along the y-axis to see if the two sides match up.	If replacing x with $-x$ results in the same equation.
origin		Turn the paper upside down to see if it looks the same.	If replacing y with $-y$ and x with $-x$ results in the same equation.

Example 3: Determine the symmetry for each equation:

A) $y = x^3 + 4x$

B) $x^6 + y^2 = 6$

C) $y = x + 1$

Even Functions: All exponents are even numbers. Have y-axis symmetry.

Odd Functions: All exponents are odd numbers. Have origin symmetry.

Example 4: Determine if the function is even, odd, or neither:

A) $y = 8x^4 - 3x^2 + 13$

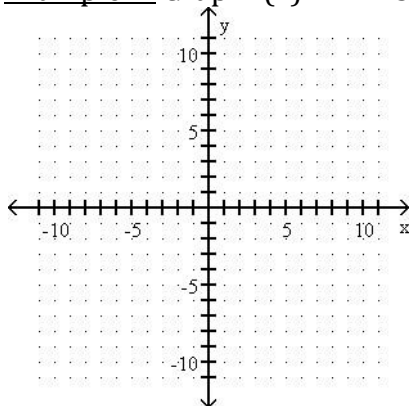
B) $y = 6x^3 - 9x$

C) $y = 3x^2 + 5x$

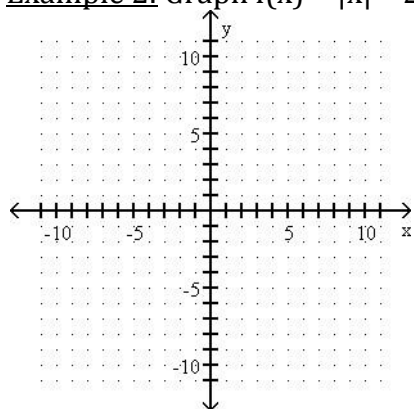
Section 2.2: Vertical and Horizontal Shifts

Vertical Shift: The graph of $y = f(x)$ is moved up c units to get $y = f(x) + c$.

Example 1: Graph $f(x) = x^2 + 3$

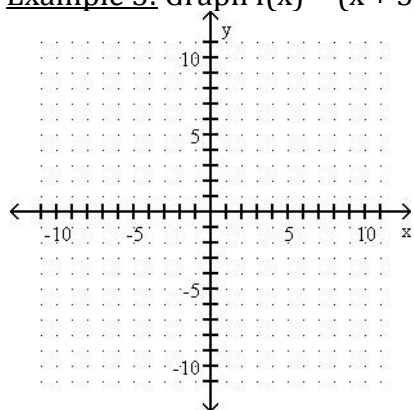


Example 2: Graph $f(x) = |x| - 2$

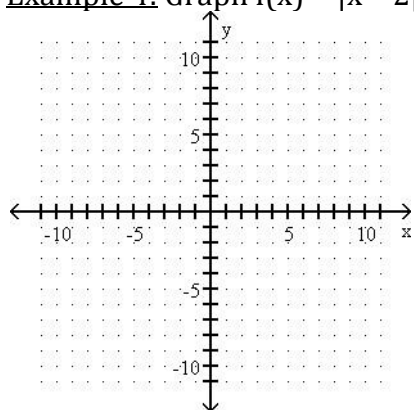


Horizontal Shift: The graph of $y = f(x)$ is moved *left* b units to get $y = f(x + b)$

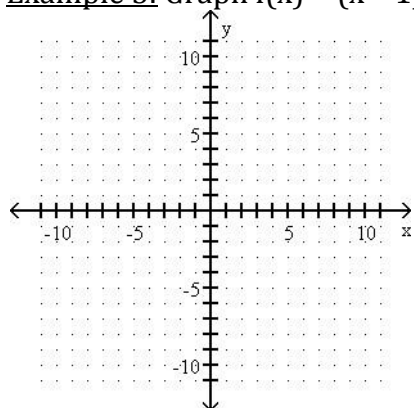
Example 3: Graph $f(x) = (x + 3)^2$



Example 4: Graph $f(x) = |x - 2|$



Example 5: Graph $f(x) = (x - 1)^3 + 4$

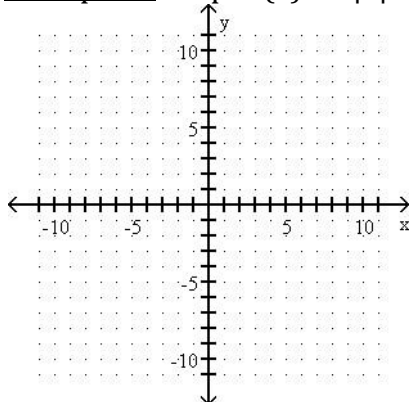


Section 2.3: Stretching, Shrinking, and Reflecting Graphs

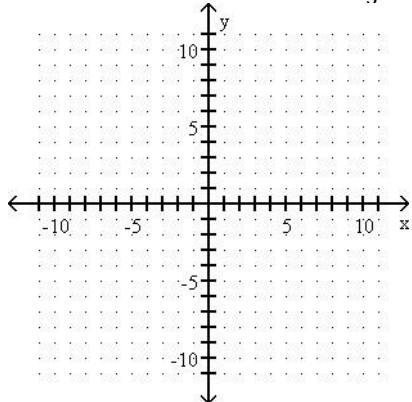
Vertical Stretch: The graph of $y = f(x)$ is stretched by a factor of a to get $y = a \cdot f(x)$ if $a > 1$.

Vertical Shrink: The graph of $y = f(x)$ is shrunk by a factor of a to get $y = a \cdot f(x)$ if $0 < a < 1$.

Example 1: Graph $f(x) = 2|x|$



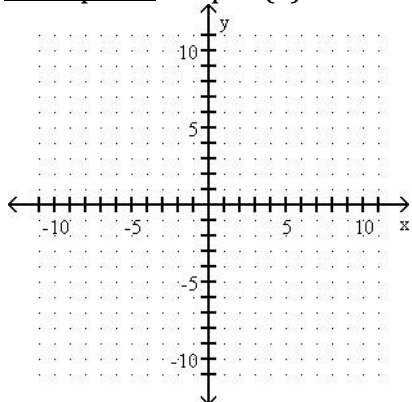
Example 2: Graph $f(x) = \frac{1}{3}x^2$



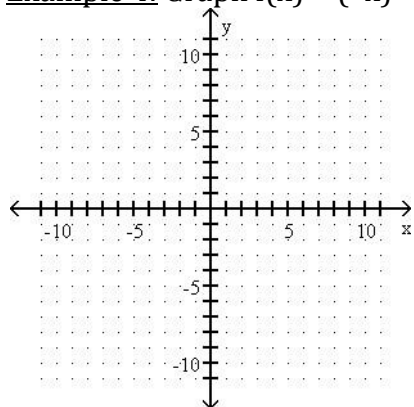
X-Axis Reflection: The graph of $y = f(x)$ is flipped over the x-axis to get $y = -f(x)$.

Y-Axis Reflection: The graph of $y = f(x)$ is flipped over the y-axis to get $y = f(-x)$.

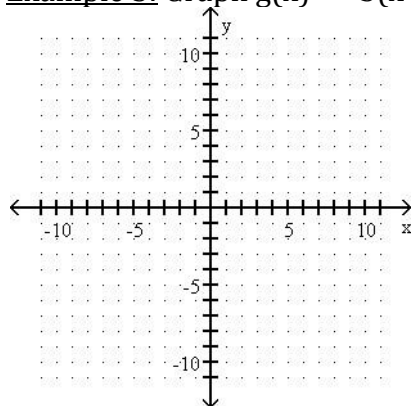
Example 3: Graph $f(x) = -x^2$



Example 4: Graph $f(x) = (-x)^2$

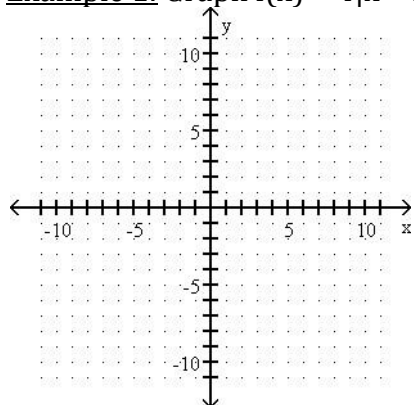


Example 5: Graph $g(x) = -3(x - 4)^3 + 5$



Section 2.4: Absolute Value Functions

Example 1: Graph $f(x) = 4|x + 2| - 5$



Absolute Value Equations: Rewrite $|x| = a$ as $x = a$ or $x = -a$, assuming $a \geq 0$.

Example 2: Solve:

$$|2x + 1| = 7$$

Example 3: Solve:

$$|4 - 3x| - 6 = 2$$

Isolate the absolute value first

Example 4: Solve:

$$|3x + 5| = 0$$

Example 5: Solve:

$$|x + 2| = -5$$

Example 6: Solve:

$$|x + 6| = |2x - 3|$$

Absolute Value Inequalities:

Case 1: $|x| < a$ Rewrite as: $-a < x < a$
In interval notation, you will get: $(-a, a)$

Case 2: $|x| \leq a$ Rewrite as: $-a \leq x \leq a$
In interval notation, you will get: $[-a, a]$

Case 3: $|x| > a$ Rewrite as: $x > a$ or $x < -a$
In interval notation, you will get: $(-\infty, -a) \cup (a, \infty)$

Case 4: $|x| \geq a$ Rewrite as: $x \geq a$ or $x \leq -a$
In interval notation, you will get: $(-\infty, -a] \cup [a, \infty)$

Example 7: Solve and write your answer in interval notation:

$$|x - 6| < 2$$

Example 8: Solve and write your answer in interval notation:

$$|5x + 1| + 1 \leq 10$$

Example 9: Solve and write your answer in interval notation:

$$|x - 3| > 7$$

Example 10: Solve and write your answer in interval notation:

$$|2x + 9| - 5 \geq 3$$

Example 11: Solve and write your answer in interval notation:

$$|x + 2| > -2$$

Example 12: Solve and write your answer in interval notation:

$$|x + 2| < -2$$

Section 2.5: Piecewise-Defined Functions

Example 1: $f(x) = \begin{cases} x + 2, & \text{if } x \leq 0 \\ \frac{1}{2}x^2, & \text{if } x > 0 \end{cases}$

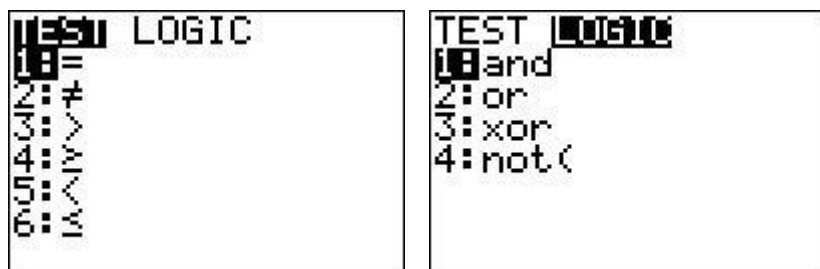
A) Evaluate $f(-3)$

B) Evaluate $f(3)$

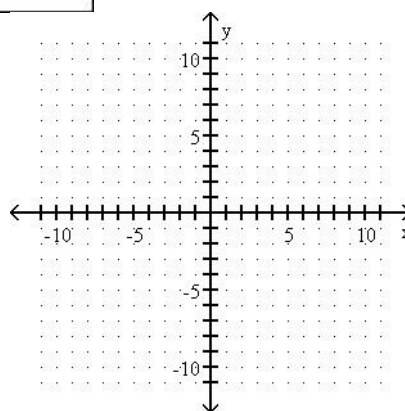
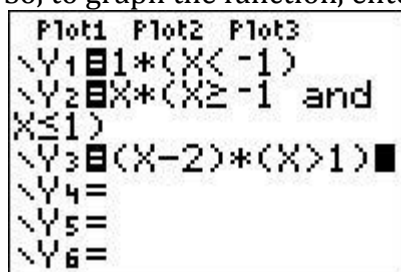
C) Evaluate $f(0)$

Example 2: Graph $f(x) = \begin{cases} 1, & \text{if } x < -1 \\ x, & \text{if } -1 \leq x \leq 1 \\ x - 2, & \text{if } x > 1 \end{cases}$

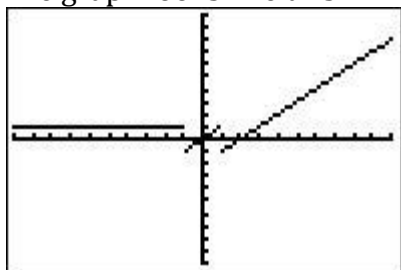
A piecewise function has to be entered as more than one graph and uses inequalities, sometimes including the 'and' value. Inequalities can be found by pressing 2ND and then MATH. 'and' can be found by pressing 2ND, then MATH, then going right once to LOGIC.



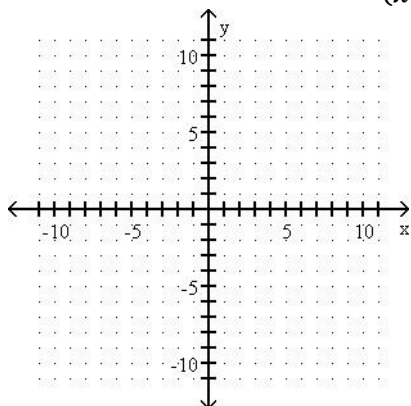
So, to graph the function, enter it under y= like this:



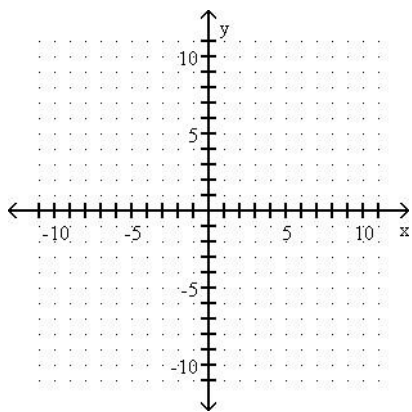
The graph looks like this:



Example 3: Graph: $f(x) = \begin{cases} x - 1, & \text{if } x < 2 \\ x + 1, & \text{if } x \geq 2 \end{cases}$



Example 4: Graph: $f(x) = \begin{cases} x - 2, & \text{if } x \leq -1 \\ x^2, & \text{if } -1 < x < 2 \\ \sqrt{x}, & \text{if } x \geq 2 \end{cases}$



Section 2.6: Operations and Composition

Operations on Functions:

1. Addition: $(f + g)(x) = f(x) + g(x)$
2. Subtraction: $(f - g)(x) = f(x) - g(x)$
3. Multiplication: $(fg)(x) = f(x)g(x)$
4. Division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.

Example 1: For the functions $f(x) = x^2 + 1$ and $g(x) = 3x + 5$, find:

A) $(f + g)(x)$

B) $(f - g)(x)$

C) $(fg)(x)$

D) $\left(\frac{f}{g}\right)(x)$

Difference Quotient: $\frac{f(x+h)-f(x)}{h}$ (you don't need to memorize this formula)

Example 2: Find the difference quotient for $f(x) = 3x + 2$

Example 3: Find the difference quotient for $f(x) = 2x^2 - 4x + 5$

Composition: The composition function is defined as: $(f \circ g)(x) = f(g(x))$

Example 4: $f(x) = 4x + 1$, $g(x) = 3x - 5$

A) Find $(f \circ g)(x)$

B) Find $(g \circ f)(x)$

Example 5: $f(x) = 2x - 6$, $g(x) = \frac{4}{x-1}$

A) Find $(f \circ g)(x)$

B) Find $(g \circ f)(x)$

Example 6: $f(x) = 6x^2 + 7x$, $g(x) = 3x - 8$

A) Find $(f \circ g)(-2)$

B) Find $(g \circ f)(4)$

Section 3.1: Complex Numbers

The imaginary number i : $i = \sqrt{-1}$ and $i^2 = -1$
 $\sqrt{-a} = \sqrt{a}i$ or $i\sqrt{a}$

Example 1: Simplify the radical:

A) $\sqrt{-36}$

B) $\sqrt{-5}$

C) $-\sqrt{-50}$

D) $\sqrt{-7}\sqrt{-7}$

E) $\sqrt{27}\sqrt{-3}$

F) $\frac{\sqrt{-125}}{\sqrt{5}}$

G) $\frac{\sqrt{-8}}{\sqrt{-2}}$

Complex Number: Any number in the form $a + bi$, where a is the real part and b is the imaginary part.

Example 2: Add:

$$(2 + 3i) + (-4 + 5i)$$

Example 3: Subtract:

$$(3 - 5i) - (-7 + i)$$

Example 4: Multiply:

$$(6 + 5i)(3 + 7i)$$

The conjugate of $a + bi$ is $a - bi$.

Example 5: Multiply $8 - 4i$ by its conjugate.

The denominator *can't* have a 0, a radical, or an i . To rationalize, multiply by the conjugate of the denominator.

Example 6: Rationalize:

A) $\frac{2+i}{1-i}$

B) $\frac{3+4i}{5-7i}$

Powers of i :

i	$i^5 = i^1$	$i^9 = i^1$	and so on
$i^2 = -1$	$i^6 = i^2 = -1$	$i^{10} = i^2 = -1$	
$i^3 = -i$	$i^7 = i^3 = -i$	$i^{11} = i^3 = -i$	
$i^4 = 1$	$i^8 = i^4 = 1$	$i^{12} = i^4 = 1$	

Example 7: Simplify the power of i :

A) i^{37}

Is the exponent divisible by 4? No.

What is the largest number that is divisible by 4 and is smaller than the exponent? 36

$$i^{37} = i^{36+1} = i^{36}i^1 = i^{(4)(9)}i^1 = (i^4)^9i^1 = 1^9i^1 = i$$

B) i^{98}

C) i^{203}

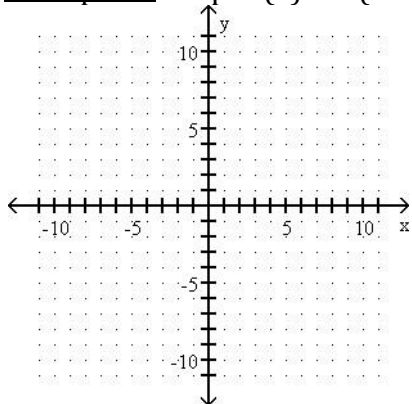
D) i^{500}

Section 3.2: Quadratic Functions and Graphs

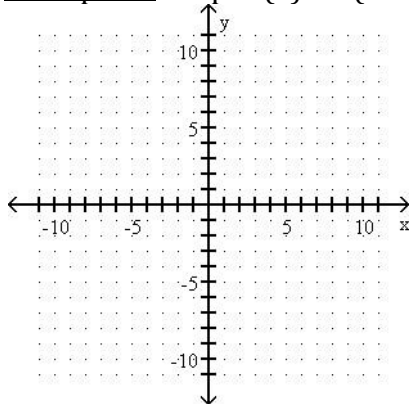
Quadratic Function: of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

Vertex-Form of the Quadratic Function: $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex. The parabola opens up if $a > 0$ and the parabola opens down if $a < 0$.

Example 1: Graph $f(x) = -(x + 3)^2 + 1$. Label the vertex.



Example 2: Graph $f(x) = 2(x - 1)^2 - 4$. Label the vertex.



Vertex Formula: The vertex of the parabola $f(x) = ax^2 + bx + c$ is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Example 3: Find the vertex of the parabola $f(x) = 2x^2 + 4x + 6$

Height of a Projectile: $s(t) = -16t^2 + v_0t + s_0$, where v_0 is the initial velocity and s_0 is the initial height.

Example 4: A ball is thrown directly into the air from a height of 100 feet with an initial speed of 80 feet per second. When does the ball reach its maximum height? What is the maximum height?

Section 3.3: Quadratic Equations and Inequalities

Zero Factor Theorem: if $ab = 0$, then $a = 0$ or $b = 0$.

Example 1: Solve:

$$x^2 - x - 6 = 0$$

Example 2: Solve:

$$2x^2 - 4x - 16 = 0$$

Example 3: Solve:

$$x^2 - 6x + 9 = 0$$

Square Root Property: if $x^2 = k$, then $x = \pm\sqrt{k}$

Example 4: Solve:

$$2x^2 = -10$$

Example 5: Solve:

$$(x - 1)^2 = 49$$

Quadratic Formula: the solution to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 6: Solve:

$$x^2 - 4x = -2$$

Example 7: Solve:

$$2x^2 - x + 4 = 0$$

Example 8: Solve and write your answer in interval notation:

$$x^2 - x - 12 < 0$$

Step 1: Solve as if equal.

Step 2: make a number line and test intervals.

Step 3: Graph on calculator to check and write answer in interval notation.

Example 9: Solve and write your answer in interval notation:

$$2x^2 \geq -5x + 12$$

Example 10: Solve and write your answer in interval notation:

$$x^2 - 4x + 2 \leq 0$$

Example 11: Solve the formula $A = \frac{\pi d^2}{4}$ for d.

Example 12: Solve the formula $rt^2 - st = k$ for t.

Section 3.4: Applications of Quadratic Functions

Example 1: A farmer wishes to enclose a rectangular region. He has 120 feet of fencing and plans to use his barn as one side of the enclosure. What is the maximum area that he can enclose?

Example 2: In a right triangle, the longer leg is 20 meters longer than twice the length of the shorter leg. The length of the hypotenuse is 30 meters longer than twice the length of the shorter leg. Find the lengths of all three sides of the right triangle.

Example 3: A machine produces rectangular sheets of metal with the length three times the width. Equal-sized squares measuring 5 inches per side are cut from the corners so that the resulting piece of metal can be folded into an open-top box. If the volume of the box is 1435 cubic inches, what are the dimensions of the original piece of metal?

Section 3.5: Higher Degree Polynomial Functions

Degree of a Polynomial: The largest exponent.

End Behavior of Graphs of Polynomial Functions: For a polynomial with a leading term ax^n :

If n is even and $a > 0$



If n is even and $a < 0$



If n is odd and $a > 0$



If n is odd and $a < 0$



Example 1: Determine the end behavior for each polynomial:

A) $f(x) = x^4 - x^2 + 5x - 4$

B) $g(x) = -x^6 + x^2 - 3x - 4$

C) $h(x) = 3x^3 - x^2 + 2x - 4$

D) $k(x) = -x^7 + x - 4$

Section 3.6: Topics in the Theory of Polynomial Functions I

Example 1: Use synthetic division to find $(5x^3 - 6x^2 - 28x + 8) \div (x + 2)$

Example 2: Divide: $(6x^4 - 3x^3 + 17x^2 - x + 2) \div (x - 1)$

Example 3: Divide: $(x^3 + 8) \div (x + 2)$

Remainder Theorem: If the polynomial function $P(x)$ is divided by $x - k$, the remainder is equal to $P(k)$.

Example 4: $P(x) = -x^4 + 3x^2 - 4x - 5$. Find $P(-2)$.

Factor Theorem: The polynomial function $P(x)$ has a factor of $x - k$ if and only if $P(k) = 0$. In other words, k is a zero of P if dividing $P(x)$ by $x - k$ has a remainder of 0.

Example 5: Is 2 a zero of the polynomial $P(x) = x^3 - 4x^2 + 9x - 10$?

Example 6: Is $x - 1$ a factor of the polynomial $P(x) = 2x^4 + 3x^2 - 5x + 7$?

Example 7: Divide: $(5x^3 - 4x^2 + 7x - 2) \div (x^2 + 1)$

You can only use synthetic division if the divisor is of the form $x - k$. Here, you must use long division.

Section 3.7: Topics in the Theory of Polynomials II

Conjugate Zeros Theorem: If $a + bi$ is a zero of a polynomial, then its conjugate $a - bi$ is also a zero.

Fundamental Theorem of Algebra: Every polynomial function of degree 1 or more has at least one complex zero.

Number of Zeros Theorem: A polynomial function of degree n has at most n distinct, complex zeros.

Example 1: Find all the zeros of the polynomial $P(x) = 2x^3 - 5x^2 - x + 6$ given that -1 is a zero.

Example 2: Find all the zeros of the polynomial $P(x) = x^3 + x^2 - 8x - 12$ given that -2 is a zero.

Rational Zeros Theorem: If p/q is a zero of the polynomial function $P(x)$, then p is a factor of the constant term and q is a factor of the leading coefficient.

Example 3: Find all the zeros of $P(x) = 6x^3 - 5x^2 - 7x + 4$. Then factor.

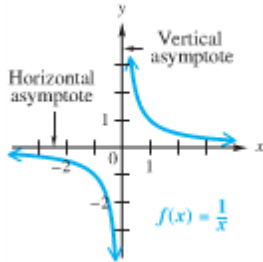
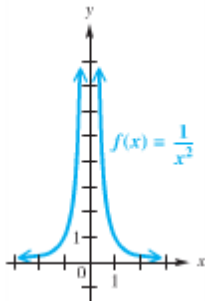
Example 5: Find all the zeros of $P(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$

Example 6: Find all the zeros of $P(x) = 6x^4 - 5x^3 - 11x^2 + 10x - 2$

Example 7: Find all the rational zeros of $P(x) = x^3 + 7$

Section 4.1: Rational Functions and Graphs

Rational Function: $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and $q(x) \neq 0$.

Name and Equation	Graph	Domain and Range
Reciprocal Function $f(x) = \frac{1}{x}$		Domain: $x \neq 0$ Range: $y \neq 0$
Rational Function $f(x) = \frac{1}{x^2}$		Domain: $x \neq 0$ Range: $y > 0$

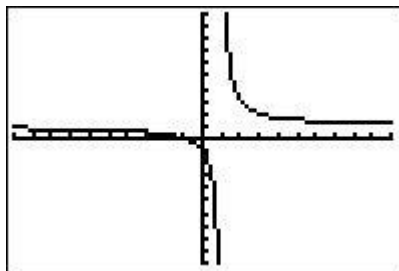
Example 1: Graph $f(x) = \frac{x+1}{x-1}$

To graph on the calculator, put parentheses around the entire numerator and the entire denominator.

```

Plot1 Plot2 Plot3
Y1=(X+1)/(X-1)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=

```



X	Y1
-3	.5
-2	.33333
-1	0
0	-1
1	ERROR
2	2
3	2

X = -3

Example 2: Graph $f(x) = \frac{1}{(x-1)^2}$

Section 4.2: More on Rational Functions and Graphs

Vertical Asymptote: where the denominator equals zero.

Example 1: Find the Vertical Asymptote(s):

A) $f(x) = \frac{x+7}{x-2}$

B) $f(x) = \frac{x+1}{x^2-5x+6}$

Horizontal and Oblique Asymptotes: Find the degree of the numerator and the denominator.

Case 1: The numerator and denominator are the same degree. HA is the fraction of leading coefficients. No OA.

Case 2: The denominator has a higher degree than the numerator. HA is $y = 0$. No OA.

Case 3: The numerator is exactly one degree higher than the denominator. OA is found by performing synthetic division and ignoring the remainder. No HA.

Case 4: The numerator is two or more degrees higher than the denominator. No HA, no OA.

Example 2: Find the Horizontal or Oblique Asymptote:

A) $f(x) = \frac{3x+2}{2x-7}$

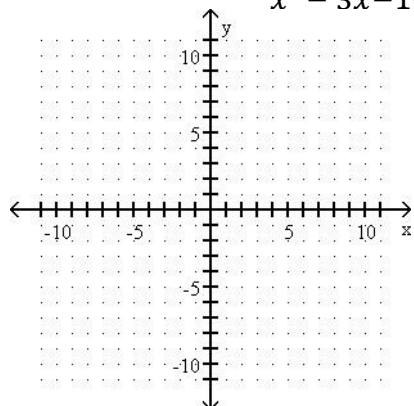
B) $f(x) = \frac{3x+2}{2x^2-7}$

C) $f(x) = \frac{x^2+1}{x-2}$

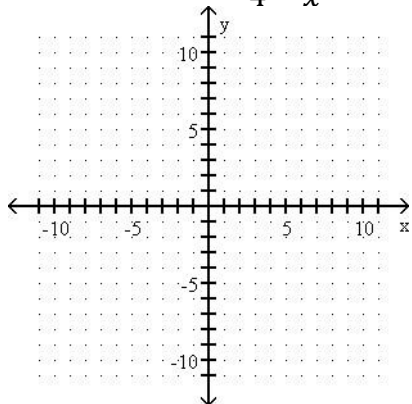
D) $f(x) = \frac{3x^3+2}{2x-7}$

Example 3: Graph and label asymptotes:

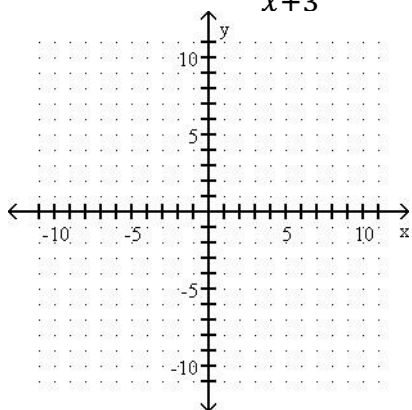
A) $f(x) = \frac{4x^2 + 4x - 24}{x^2 - 3x - 10}$



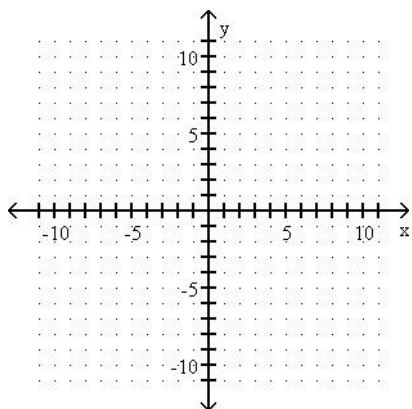
B) $f(x) = \frac{x}{4 - x^2}$



C) $f(x) = \frac{x^2 + 1}{x + 3}$



D) $f(x) = \frac{x^2 - 4}{x - 2}$



Section 4.3: Rational Equations and Inequalities

Example 1: Solve:

$$\frac{x+2}{2x+1} = 1$$

To solve, multiply both sides by the LCD. When you're done, check to be sure you haven't divided by zero.

Example 2: Solve:

$$\frac{x}{x-2} + \frac{1}{x+2} = \frac{8}{x^2-4}$$

Example 3: Solve and write your answer in interval notation:

$$\frac{x+1}{x-4} \geq 0$$

Step 1: Solve as if equal.

Step 2: Find the domain (where the denominator isn't equal to zero).

Step 3: Make a number line and test intervals.

Step 4: Check by graphing with the calculator and write answer in interval notation.

Example 4: Solve and write your answer in interval notation:

$$\frac{x-1}{x+2} \leq 0$$

Inverse Variation: “y varies inversely as the nth power of x.” Use $y = \frac{k}{x^n}$

Example 5: The intensity of light varies inversely as the distance squared. At a distance of 3 meters, the intensity is 88 watts per square meter. Find the intensity when the distance is 2 meters.

Joint Variation: “y varies jointly as the nth power of x and the mth power of z.” Use $y = kx^n z^m$

Example 6: The volume varies jointly as the 1.12 power of height and the 1.98 power of diameter. If the volume is 25.14 cubic feet when the diameter is 13.8 inches and the height is 64 feet, find the volume when the diameter is 11 inches and the height is 47 feet.

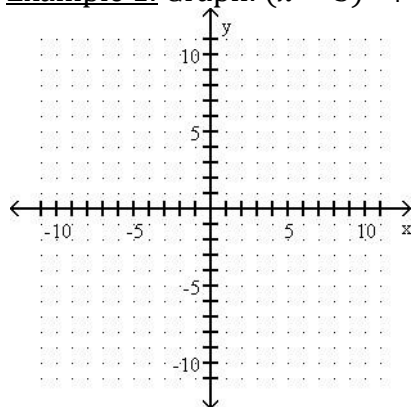
Rate of Work: If a task can be completed in x amount of time, the rate of work is $\frac{1}{x}$

Example 7: It takes machine B one hour less to complete a task than machine A. Working together, it takes 1.2 hours. How long does it take machine A?

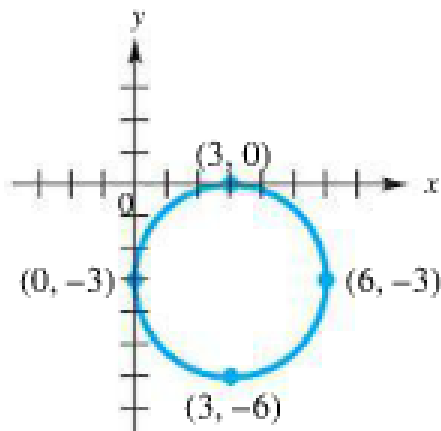
Section 6.1: Circles and Parabolas

Center-Radius Form of the Equation of a Circle: $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius.

Example 1: Graph: $(x - 5)^2 + (y + 2)^2 = 9$

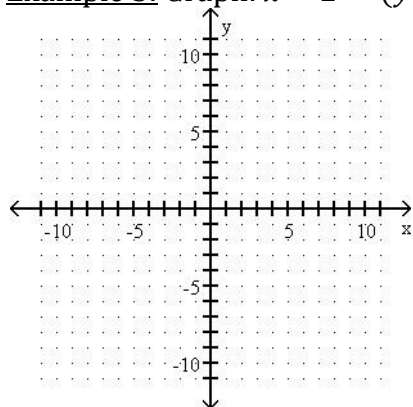


Example 2: Find the equation of the circle:

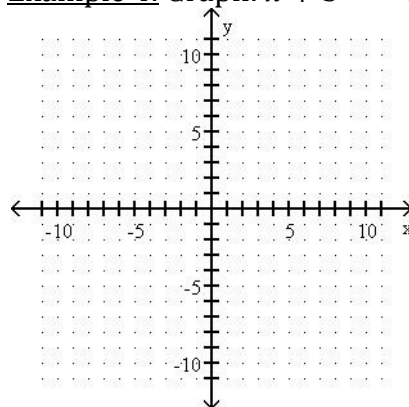


Parabola with Horizontal Axis: $x - h = a(y - k)^2$, where (h, k) is the vertex. If $a > 0$, the parabola opens to the right. If $a < 0$, the parabola opens to the left.

Example 3: Graph: $x - 2 = (y + 1)^2$



Example 4: Graph: $x + 3 = -(y - 4)^2$



Section 6.2: Ellipses and Hyperbola

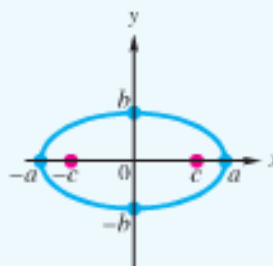
Ellipse: The set of all points in a plane, the sum of whose distances from 2 fixed points (foci) is constant.

Standard Forms of Equations for Ellipses

The ellipse with center at the origin and equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

has vertices $(\pm a, 0)$, endpoints of the minor axis $(0, \pm b)$, and foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

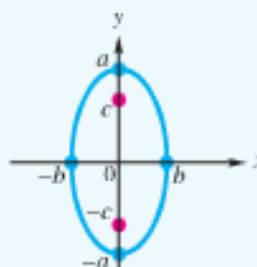


Major axis on x-axis

The ellipse with center at the origin and equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b > 0)$$

has vertices $(0, \pm a)$, endpoints of the minor axis $(\pm b, 0)$, and foci $(0, \pm c)$, where $c^2 = a^2 - b^2$.



Major axis on y-axis

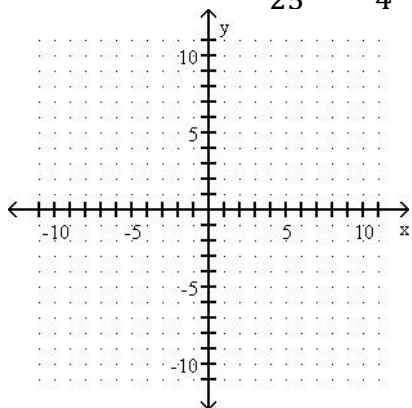
Standard Forms for Ellipses Centered at (h, k)

An ellipse with center at (h, k) and either a horizontal or vertical major axis satisfies one of the following equations, where $a > b > 0$ and $c^2 = a^2 - b^2$ with $c > 0$:

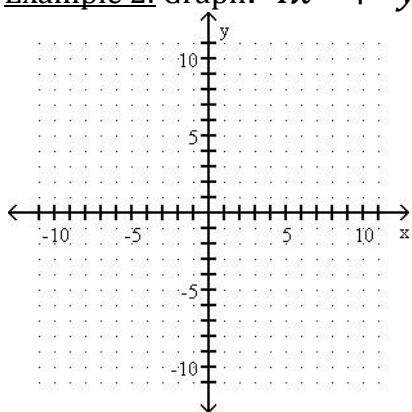
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{Major axis: horizontal; foci: } (h \pm c, k); \text{ vertices: } (h \pm a, k)$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \quad \text{Major axis: vertical; foci: } (h, k \pm c); \text{ vertices: } (h, k \pm a)$$

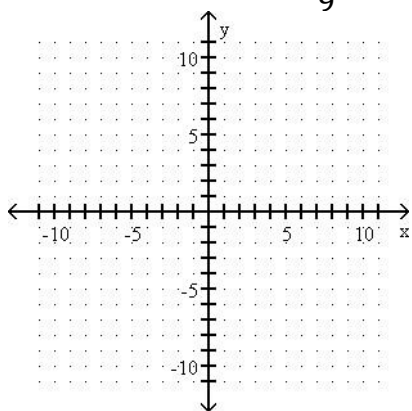
Example 1: Graph: $\frac{x^2}{25} + \frac{y^2}{4} = 1$



Example 2: Graph: $4x^2 + y^2 = 64$



Example 3: Graph: $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{16} = 1$



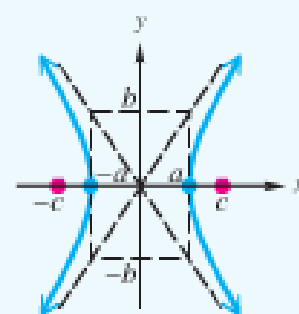
Hyperbola: The set of all points in a plane such that the absolute value of the difference of the distances from 2 fixed points (foci) is constant

Standard Forms of Equations for Hyperbolas

The hyperbola with center at the origin and equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has vertices $(\pm a, 0)$, asymptotes $y = \pm \frac{b}{a}x$, and foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

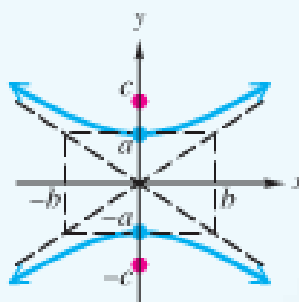


Transverse axis on x-axis

The hyperbola with center at the origin and equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has vertices $(0, \pm a)$, asymptotes $y = \pm \frac{a}{b}x$, and foci $(0, \pm c)$, where $c^2 = a^2 + b^2$.



Transverse axis on y-axis

Standard Forms for Hyperbolas Centered at (h, k)

A hyperbola with center (h, k) and either a horizontal or vertical transverse axis satisfies one of the following equations, where $c^2 = a^2 + b^2$.

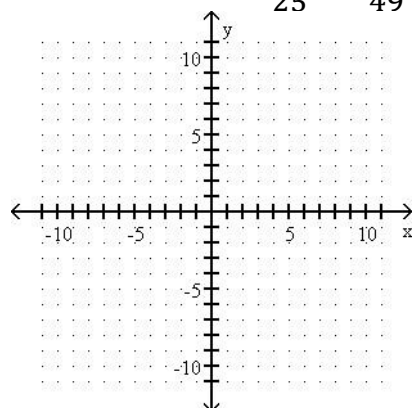
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Transverse axis: horizontal;
vertices: $(h \pm a, k)$; foci: $(h \pm c, k)$;
asymptotes: $y = \pm \frac{b}{a}(x - h) + k$

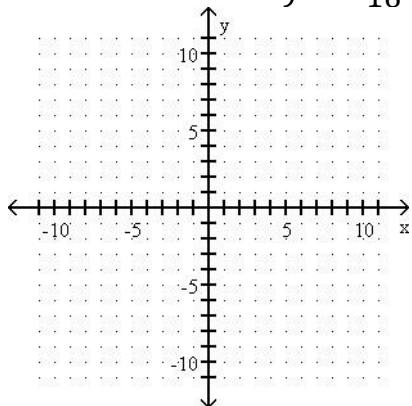
$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

Transverse axis: vertical;
vertices: $(h, k \pm a)$; foci: $(h, k \pm c)$;
asymptotes: $y = \pm \frac{a}{b}(x - h) + k$

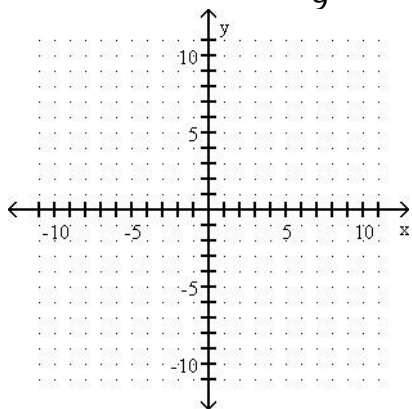
Example 4: Graph: $\frac{x^2}{25} - \frac{y^2}{49} = 1$



Example 5: Graph: $\frac{y^2}{9} - \frac{x^2}{16} = 1$



Example 6: Graph: $\frac{(y+2)^2}{9} - \frac{(x+3)^2}{4} = 1$



Section 6.3: Summary of Conics

Conic	Equation	Characteristic
Parabola	$(x - h)^2 = y - k$	Only one variable is squared
Circle	$(x - h)^2 + (y - k)^2 = r^2$ $\frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$	The denominators are the same
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	One denominator is larger
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Minus sign

To determine which conic the equation is, write in one of the standard forms listed above.

Example 1: Determine which conic the equation describes.

A) $x^2 = 25 + 5y^2$

B) $y + x^2 = 7$

C) $y^2 = 64 - 4x^2$

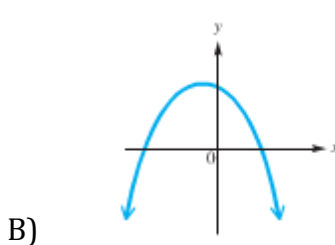
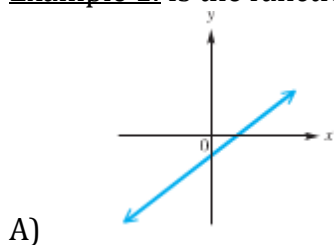
D) $x^2 = 9 - y^2$

Section 5.1: Inverse Functions

One-to-One Function: if $a \neq b$ implies that $f(a) \neq f(b)$.

Horizontal Line Test: If every horizontal line intersects the graph at no more than one point, then the function is one-to-one.

Example 1: Is the function one-to-one?



Inverse Function: Let f be one-to-one. Then g is the inverse function of f if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

f^{-1} is the notation for the inverse function.

f^{-1} must also be one-to-one.

Inverse functions are symmetric about the line $y = x$.

The domain of f is the same as the range of f^{-1} .

The range of f is the same as the domain of f^{-1} .

Example 2: Determine whether the two functions are inverses of each other.

A) $f(x) = x^3 - 1, g(x) = \sqrt[3]{x + 1}$

If both compositions equals x , then the answer is yes.

B) $f(x) = 2x + 3, g(x) = 2x - 3$

Example 3: Find f^{-1} :

A) $f(x) = 2x + 5$

Step 1: interchange the variables.

Step 2: Solve for y.

B) $f(x) = (x - 2)^5$

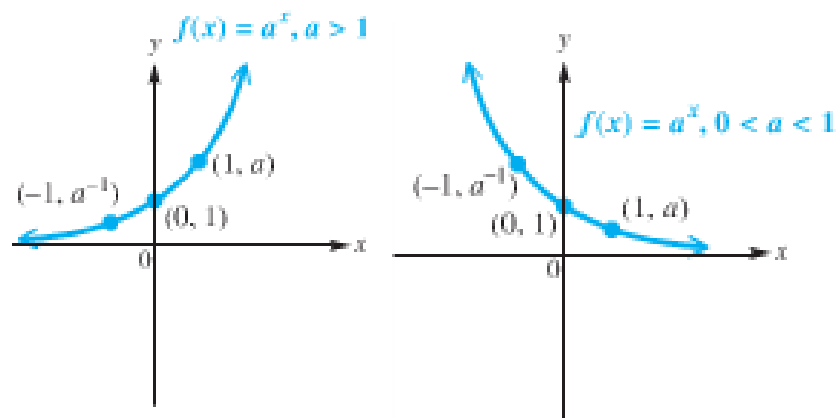
C) $f(x) = x^2 + 3$

D) $f(x) = \frac{2x}{x-1}$

E) $f(x) = \sqrt{x+3}$

Section 5.2: Exponential Functions

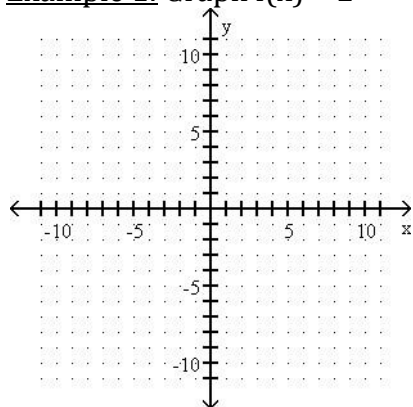
Exponential Function: of the form $f(x) = a^x$ where a is the base, $a > 0$, $a \neq 1$.



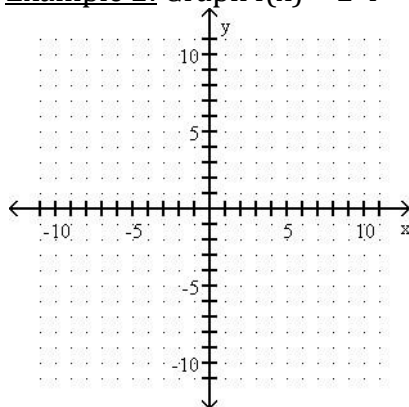
Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Example 1: Graph $f(x) = 2^x$



Example 2: Graph $f(x) = 2 \cdot 4^{-x+1} - 3$



Exponent Property: $a^x = a^y$ if and only if $x = y$.

Example 3: Solve:

$$25^x = 125$$

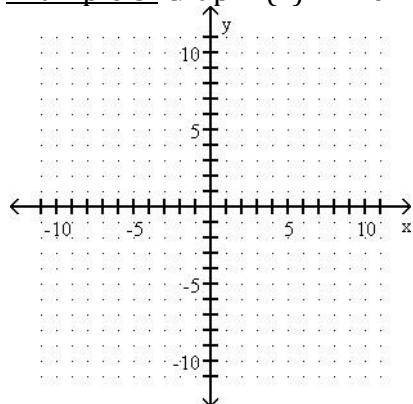
Rewrite each side with the same base.

Example 4: Solve:

$$\left(\frac{1}{3}\right)^x = 81$$

The number e: Approximately 2.718

Example 5: Graph $f(x) = -2e^{x-8} + 4$



Section 5.3: Logarithms

Logarithm: $y = \log_a x$ "log base a of x" means $a^y = x$, where $a > 0$, $a \neq 1$, and $x > 0$.

Example 1: Rewrite:

A) $2^3 = 8$

B) $\left(\frac{1}{2}\right)^{-4} = 16$

C) $\log_5 25 = 2$

D) $\log_3 81 = 4$

Example 2: Solve:

$$\log_x 16 = 4$$

Rewrite as an exponential then solve.

Example 3: Solve:

$$\log_4 x = \frac{3}{2}$$

Example 4: Solve:

$$\log_8 4 = x$$

Common Log: $\log x = \log_{10} x$

Natural Log: $\ln x = \log_e x$

Example 5: Evaluate:

A) $\log 12$

B) $\ln 7$

Log Properties:

1. $\log_a 1 = 0$

Example 6: Rewrite using log properties: $\log_{17} 1$

2. $\log_a a = 1$

Example 7: Rewrite using log properties: $\log_{30} 30$

3. Product Rule: $\log_a xy = \log_a x + \log_a y$

Example 8: Expand: $\log_2 3x$

Example 9: Write as a single log: $\log_3 4 + \log_3 y$

Example 10: Expand: $\log_4(x + 5)$

4. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

Example 11: Expand: $\log_5 \frac{3}{x}$

Example 12: Write as a single log: $-\log_6 y + \log_6 10$

Example 13: Expand: $\log_7(y - 9)$

5. Power Rule: $\log_a x^r = r \log_a x$

Example 14: Expand: $\log_8 x^2$

Example 15: Write as a single log: $4 \log_7 y$

6. $\log_a a^r = r$

Example 16: Rewrite using log properties: $\log 10^x$

7. $a^{\log_a r} = r$

Example 17: Rewrite using log properties: $e^{\ln x}$

Example 18: Expand: $\log_4 \frac{x^4 y^5}{z^6}$

Example 19: Expand: $\log_b \frac{a^7 c^8}{b^9 d^{10}}$

Example 20: Write as a single log: $2 \log_3 x - 3 \log_3 y + 4 \log_3 w - 5 \log_3 z$

Example 21: Write as a single log: $-4 \log_8 a - 7 \log_8 b$

Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$, where b is a number we choose, usually 10 or e.

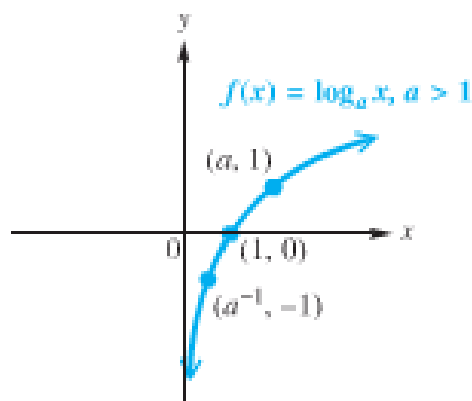
Example 12: Evaluate:

A) $\log_5 17$

B) $\log_3 27$

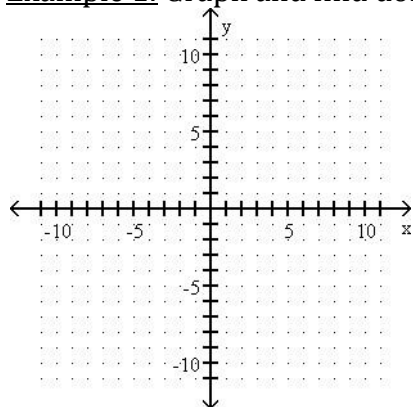
Section 5.4: Logarithm Functions

Logarithm Function: of the form $f(x) = \log_a x$, where $a > 0$, $a \neq 1$, and $x > 0$.



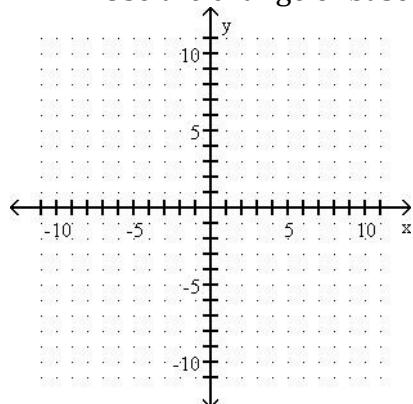
Domain: $x > 0$
 $(0, \infty)$
 Range: $(-\infty, \infty)$

Example 1: Graph and find domain: $f(x) = \ln x$



Example 2: Graph and find domain: $f(x) = \log_2(x - 1)$

Use the change of base formula.



Section 5.5: Exponential and Logarithm Equations and Inequalities

Properties Logarithmic Functions: $\log_a x = \log_a y$ if and only if $x = y$.

Example 1: Solve and round your answer to 3 decimal places:

$$7^x = 12$$

If you can't rewrite each side with the same base, take a log on both sides.

Example 2: Solve and round your answer to 3 decimal places:

$$2^{3x+1} = 3^{4-x}$$

Example 3: Solve and round your answer to 3 decimal places:

$$5^{x+2} = 4^{x+6}$$

Example 4: Solve, round your answer to 3 decimal places, and write your answer in interval notation:

$$6^x < 8$$

Step 1: Solve as if equal.

Step 2: Make a number line and test intervals.

Step 3: Check with calculator and write answer in interval notation.

Example 5: Solve, round your answer to 3 decimal places, and write your answer in interval notation:

$$4^x \geq 9$$

Example 6: Solve:

$$\log_3(x + 6) - \log_3(x + 2) = \log_3 x$$

There are 2 types of log equations. The first type has a log for every term.

Step 1: Combine logs on each side.

Step 2: Use log property to drop logs on each side. Then solve.

Step 3: Keep positive answers only.

Example 7: Solve:

$$\ln e^{\ln x} - \ln(x - 3) = \ln 2$$

Example 8: Solve:

$$\log(3x + 2) + \log(x - 1) = 1$$

The second type of log equation includes one term which is a constant.

Step 1: Combine logs

Step 2: Rewrite as an exponential equation and solve.

Step 3: Keep positive answers only.

Example 9: Solve:

$$\log x + \log x^2 = 3$$

Example 10: Solve and write your answer in interval notation:

$$\log_2(2x) \leq 5$$

Step 1: Find the domain. (what you take a log of must be > 0).

Step 2: Solve as if equal.

Step 3: Make a number line and test intervals.

Step 4: Check with calculator and write answer in interval notation.

Example 11: Solve and write your answer in interval notation:

$$\ln(x - 2) > 0$$

Section 7.1: Systems of Equations

Example 1: Solve by substitution:

$$3x + 2y = 11$$

$$-x + y = 3$$

Example 2: Solve by elimination:

$$3x - 4y = 1$$

$$2x + 3y = 12$$

Example 5: Solve using the calculator:

$$3x - y = 4$$

$$x + y = 0$$

To solve the system, you have to enter it as the 2×3 matrix $\begin{bmatrix} 3 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix}$. To do so, you have to access the matrix menu. Press 2ND then the x^{-1} key. This will get you into the matrix menu. Go right twice to EDIT and hit ENTER.



Now you need to enter your matrix. First, enter the dimensions, which in this example are 2×3 . Then enter the values of the actual matrix. When you're done, it should look like this:



Now quit the matrix menu by pressing 2ND then MODE. You'll have to go back to the matrix menu by pressing 2ND then x^{-1} . Go right once to MATH. Scroll down until you get to the option that says rref(. Note: this is different from just ref(. so make sure you get the right one. Hit enter.

```

NAMES [MATH] EDIT
6:randM(
7:augment(
8:Matr→list(
9>List→matr(
0:cumSum(
A:ref(
B:rref(

```

You will now have to go back to the matrix menu one more time by pressing 2ND then x-1. Select the matrix you edited earlier by hitting ENTER. Then hit ENTER again. Your screen should read:

```

rref([A])
[[1 0 1 ]
 [0 1 -1]]

```

The answer is the right-most column in the answer matrix. In this example, the answer is (1, -1).

Example 6: Solve using the calculator:

$$\begin{aligned} -2x - 6y &= 18 \\ -3x + 5y &= -29 \end{aligned}$$

Example 7: Solve the nonlinear system:

$$\begin{aligned} 3x^2 - 2y &= 5 \\ x + 3y &= -4 \end{aligned}$$

You can't solve a nonlinear system on the calculator. Since some terms are linear and some are nonlinear, use substitution.

Example 8: Solve:

$$y = -x^2 + 2$$

$$x - y = 0$$

Example 9: Solve:

$$x^2 + y^2 = 4$$

$$2x^2 - y^2 = 8$$

Since all the terms are nonlinear, use elimination.

Example 10: Solve:

$$x^2 + y^2 = 10$$

$$2x^2 - y^2 = 17$$

Section 7.2: Linear Equations in 3 Variables

Example 1: Solve using the calculator:

$$3x + 9y + 6z = 3$$

$$2x + y - z = 2$$

$$x + y + z = 2$$

Example 2: Solve using the calculator:

$$x + 3y + 4z = 14$$

$$2x - 3y + 2z = 10$$

$$3x - y + z = 9$$